CONSTRUCTION SCHEMES: TRANSFERRING STRUCTURES FROM ω TO ω_1

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Abstract. A structural analysis of construction schemes is developed. That analysis is used to give simple and new constructions of combinatorial objects which have been of interest to set theorists and topologists. We then continue the study of capturing axioms associated to construction schemes. From them, we deduce the existence of several uncountable structures which are known to be independent from the usual axioms of Set Theory. Lastly, we prove that the capturing axiom $FCA(part)$ is implied by Jensen's $\diamondsuit$ principle.

The purpose of this paper is to continue developing a technique introduced by the third author in [77] for constructing mathematical objects of cardinality ω_1, the first uncountable cardinal. Towards explaining the nature of this construction scheme, suppose we want to build a particular structure of cardinality ω_1. Probably the most straight forward approach is to build such object by countable approximations in ω_1 many steps. Typical examples are the construction of an Aronszajn tree from [34] or [28] and the construction of a Hausdorff gap from [62], [30] or [86] [1]. The topic of the present work is on the method of *construction and capturing schemes*. In our approach, the desired structure is not built from its countable substructures, but rather from its finite ones. Also, the construction takes only countably many steps. The approach is based on amalgamations of (many) isomorphic finite structures a method that has quite different flavour than the method used in typical constructions by countable approximations.

The idea of building an uncountable structure by finite approximations is not new. For example, this approach was previously performed with R. Jensen's gap 1-morasses (see [10]) and its simplification by D. Velleman (see [87]). Their motivation was to construct model-theoretic structures by amalgamating *two* finite structures at a given stage. This is greatly improved using construction schemes where many finite isomorphic structures are amalgamated at a given stage.

Roughly speaking, a construction scheme is a collection of finite sets with strong coherent properties. These properties are allowing us recursively follow the construction scheme and perform amalgamations of several (not just two) isomorphic finite structures. As in the case

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[1] Of course, there are also other constructions of these objects that do not rely in a recursions of size ω_1, see [74].

of Jensen's and Velleman's morasses, the existence of construction schemes on ω_1 can be obtained just from ZFC, without the need to appeal to any extra axiom. However, sometimes in applications of our approach, we need that the construction scheme have further "capturing properties" which informally means that it can "capture" parts of an uncountable set in "isomorphic positions". This will be formally introduced and explained in Section 5. For the existence of a *capturing construction scheme* we need to go beyond ZFC. It is worth pointing out that the study of construction schemes was originally motivated by problems concerning metric structures such as normed spaces and Boolean algebras. The paper [77] contains several applications of capturing construction schemes to functional analysis and topology. The general theory was further developed in [31], [42] and [43].

In the present paper, we construct a large amount of combinatorial objects that have been of interest to set theorists and topologists. The constructions developed here are rather different from the original constructions. It is often the case that using construction or capturing schemes greatly simplifies the previously known constructions (for example, 5.23). In other cases, there was no direct construction previously known (see Theorem 4.17).

To follow this paper no previous knowledge of construction and capturing schemes is needed. In fact, we strive to make this a good introduction to this fascinating topic. It is our hope to spread the interest towards readers versed in amalgamation techniques in their own fields giving them a powerful tool for building structures of their own interest.

In Section 2 we introduce the notion of construction scheme and derive its basic properties. In Section 3 we study the relation of construction schemes and the ordinal metrics studied in [74]. In Section 4 we apply construction schemes to build a large number of structures that are of interest in set theory, topology and infinite combinatorics. The constructions are mostly independent from each other, so that the reader can start from the ones that she or he finds more interest. For the convencience of the reader, we list here the constructions that appear on this sections. References for further study and historic remarks will be provided as we encounter them on the Section 4:

(1) **Countryman lines**. Let $(X, <)$ be a total linear order. Except for the trivial cases X^2 is not a linear order. In this way, it makes sense to ask how many chains we need to cover it. A countryman line is an uncountable linear order whose square can be covered with only countably many chains. These orders seem so paradoxical at first glance that Countryman conjectured they do not exist. However, it was first proved by Shelah that Countryman lines do exist (see [64]).
(2) **Luzin coherent family of functions.** We now look at a generalization of the Hausdorff gaps discussed previously. A luzin coherent family of functions is a coherent system of functions supported by a pretower, in which we impose a strong non triviality condition. The importance of these families is that they provide many cohomologically different gaps. They were first studied by Talayco in [70]. Later Farah proved that such families exist (see [17]). The proof of Farah is highly nonconstructive and indirect, since it appeals to Keisler's completeness Theorem. We will build such families using a construction scheme. No previous direct construction was known.

In Section 5 we study distinct notions of capturing, which are additional strong properties we may demand from a construction scheme (We do this in an axiomatic way). The existence of such schemes can not be deduced from ZFC alone. We then apply such schemes to build several combinatorial structures whose existence is known to be independent from ZFC. Once again, we will provide a brief description of the structure that we will build using capturing schemes:

(1) **Suslin trees.** A Suslin tree is an Aronszajn tree in which every antichain is countable. The Suslin Hypothesis (SH) is the statement that there are no Suslin trees. We now know that SH is independent from ZFC. A related concept, the Suslin lines, were introduced by Suslin while studying the ordering of the real numbers. Kurepa was the one to realize that there is a Suslin line if and only if there is a Suslin tree. Applications and constructions from Suslin trees are abundant in the literature. We will use capturing schemes to build two types of these trees: Coherent Suslin[2] and full (also called free) Suslin trees. These two families of trees are diametrically opposed. Forcing with a Coherent tree completely destroys the *ccc*-ness of it, while with a full Suslin tree, many of its subtrees remain *ccc*.

(2) **Suslin lower semi-lattices**. If in the definition of a Suslin tree we relax the condition of being a tree to just a being a lower semi-lattice, we get the notion of a Suslin lower semi-lattice. They were introduced by Dilworth, Odell and Sari (see [11]) in the context of Banach spaces. They were then studied by Raghavan and Yorioka (see [58]). Among other things they proved that the $\diamondsuit$-principle implies that $\mathscr{P}(\omega)$ contains a Suslin lower semi-lattice. We were able to obtain the same result from a capturing scheme.

(3) **Independent coherent family of functions**. We now return to the study of gaps that are obtained from a coherent family of functions, as we did in Theorem 4.17. However, this time we want our family of gaps to be "independent". This means that we can either fill or freeze by forcing any subfamily without filling or freezing any of the remaining gaps in the family. A similar result was obtained by Yorioka assuming the $\diamondsuit$-principle in [91] (the analogue for Suslin trees was proved by Abraham and Shelah in [3]).

(4) *ccc* **destructible** 2**-bounded coloring without injective sets**. A coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ is called 2-bounded if every color appears at most 2 times. A set $A \subseteq \omega_1$ is c-injective if no color appears twice in $[A]^2$. Galvin was the first to wonder if there is a 2-bounded coloring without an uncountable injective set. He proved that such coloring exists assuming the Countinuum Hypothesis. On the other hand, the third author proved in [80] that no such coloring exist under PFA. years later, Abraham, Cummings and Smyth proved that MA is consistent with the existence of a 2-bounded coloring without uncountable injective sets (see [2]). After hearing this result, Friedman asked for a concrete example of a 2-bounded coloring without an uncountable injective set, but that such set can be added with a *ccc* partial order. In [2] such example is constructed assuming CH and the failure of the Suslin hypothesis. We will also find an example with a capturing scheme.

In the book [83], the third author developed an oscillation theory which is based on an unbounded family of functions. A plethora of applications of this theory has been found

[2]It is actually known (see [36]) that the existence of coherent Suslin tree already implies the existence of a full one.

through the years. For example the oscillation theory is key in the proof that PFA implies that the continuum is ω_2 (see [4]). we will develop a similar theory using a capturing scheme. An important difference between the classic oscillation theory and the one from capturing schemes, is that this new one is based on a bounded family of functions. Using this new oscillation theory, we can prove the existence of the following objects:

(1) **Sixth Tukey type**. The Tukey ordering is a useful perspective to compare directed partial orders. Its purpose is to study how a directed partial order behave cofinally. It was introduced by Tukey in [85] in order to study convergence in topology. The Tukey classification of countable directed partial orders is very simple: Every countable directed partial order is Tukey equivalent to 1 or to ω. The Tukey classification of directed sets of size ω_1 becomes much more interesting. We now have at least five Tukey types: 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$. We may wonder if there is a directed partial order of size ω_1 that is not Tukey equivalent to one of these five. In [21], Isbell proved that CH entails the existence of a sixth Tukey type. This was greatly improved by the third author in [81], where he proved that CH implies that there are 2^{ω_1} distinct Tukey types. On the other hand, in the same paper he showed that PFA implies that there are no sixth Tukey types. Here, we found a sixth Tukey type from a capturing scheme.

(2) $\omega_1 \not\to (\omega_1, \omega+2)^2_2$. Given $\alpha < \omega_1$, the partition $\omega_1 \to (\omega_1, \alpha)^2_2$ means that for every $c : [\omega_1]^2 \longrightarrow 2$, either there is an uncountable 0-monochromatic set, or there is a 1-monochromatic set of order type α. A celebrated result of Erdős and Rado (extending a theorem by Dushnik, Miller and Erdős) is that $\omega_1 \to (\omega_1, \omega+1)^2_2$. We may wonder if this theorem can be improved. This turns out to be independent from ZFC. The Proper Forcing Axiom implies that $\omega_1 \to (\omega_1, \alpha)^2_2$ for every $\alpha < \omega_1$ (see [76]), while $\mathfrak{b} = \omega_1$ implies $\omega_1 \not\to (\omega_1, \omega+2)^2_2$. we will prove a similar result from our oscillation theory.

(3) **Non productivity of *ccc* partial orders**. When is the product of two *ccc* partial orders again *ccc*? This question has been of interest to set theorists for a long time. On one hand, Martin axiom implies that the product of *ccc* partial orders is *ccc*. On the other hand, the failure of the Suslin Hypothesis implies the opposite. Consistent examples of two *ccc* partial orders whose product is not *ccc* have been constructed by Galvin under CH (see [34]) and by the third author under $\mathfrak{b} = \omega_1$ in [83]. As an applications of the oscillation theory developed in the section, we encounter new situations in which there are *ccc* partial orders whose product is not *ccc*.

(4) **Suslin pretowers**. In [72], the third author developed an analogue of his oscillation theory, now this time based on non-meager towers. While studying this oscillation, Borodulin-Nadzieja and Chodounský introduced the notion of a Suslin pretower. A pretower $\mathcal{T} = \{T_\alpha : \alpha \in \omega_1\}$ is Suslin if every uncountable subset of $\mathcal{T}$ contains two pairwise $\subseteq$-incomparable elements. The existence of a Suslin pretower is independent from ZFC. It can be proved that the Open Graph Axiom forbidds the existence of such families, while a Suslin pretower can be constructed assuming $\mathfrak{b} = \omega_1$ (see [5]). We will build a Suslin pretower as an application of the oscillation theory obtained from a capturing scheme.

(5) **S-spaces**. Hereditarily separable and hereditarily Lindelöf are two properties that in some sense are dual to each other. We may wonder if they always coincide. The

question is only of interest in the realm of regular spaces. An S-space is a regular hereditarily separable, non Lindelöf space. The study of S-spaces (and the dual notion, L-spaces) used to be one of the most active areas of set-theoretic topology. S-spaces can be constructed under several set-theoretic hypothesis (like CH or the negation of the Suslin Hypothesis). It was the third author who for the first time succeeded in proving that it is consistent that there may be no S-spaces (see [83]). Here, we will apply the oscillation theory obtained from a capturing scheme to construct several S-spaces.

The following table summarizes many of the constructions related to set theory and topology which can be carried out with a capturing scheme.

Result	**Extra set-theoretic assumption**	**Result number**
Sixth Tukey type	CA_2	6.17
$\omega_1 \not\to (\omega_1, \omega+2)^2_2$	CA_2	6.15
Non productivity of *ccc*	CA_2	6.9
Suslin pretowers	CA_2	6.20
S-spaces	CA_2	6.25 and 6.34
Suslin lower semi-lattices	CA_2	5.23
Failure of $BA(\omega_1)$	CA_2	6.42
Destructible gap	$CA_2(part)$	see [43]
ccc destructible 2-bounded coloring	CA_3	5.26
Full Suslin Tree	FCA	5.20
Coherent Suslin Tree	$FCA(part)$	5.17
Independent coherent family of functions	FCA	5.32

Finally, in Section 7 we prove that capturing construction schemes exist assuming Jensen's $\diamondsuit$-principle. There is however still much to learn and understand on this topic. Some of the open problems or directions for further research will be mentioned throughout the paper.

1. Notation and preliminaries

Our notation is fairly standard. Most of the concepts used will be defined in the sections where they first appear. The undefined notions related to set theory can be found in [28] or [34]. We will try our best to point out more specific bibliography when it is necessary. Nevertheless, we will review some of the main notions used throughout the paper for the convenience of the reader. The quantifiers $\forall^\infty$ and $\exists^\infty$ should be read as *for all but finitely many* and *for infinitely many* respectively. The quantifier $\exists!$ stands for *there is a unique.*

We denote the power set of X by $\mathscr{P}(X)$. The cardinality of X is denoted by $|X|$. Given a cardinal κ, we let $[X]^\kappa$ be the set of all subsets of X of size κ and $[X]^{<\kappa}$ denotes $\bigcup_{\alpha<\kappa}[X]^\alpha$. The set $[X]^{\leq\kappa}$ is defined in a similar way. Lim denotes the set of limit ordinals strictly smaller than ω_1. By FIN(X) we mean $[X]^{<\omega}\backslash\{\emptyset\}$. FIN$(\omega)$ is just denoted as FIN. We write $X \subseteq^* Y$ if $X\backslash Y$ is finite and $X =^* Y$ if $X \subseteq^* Y$ and $Y \subseteq^* X$. Following this notation, we write $X \subsetneq^* Y$ whenever $X \subseteq^* Y$ but $X \neq^* Y$. Analogously, if f, g are two functions with domain X and an ordered set as a codomain, we write $f <^* g$ if $\{x \in X : g(x) \leq f(x)\}$ is finite. We write $f < g$ if the previous set is moreover empty.

If $(X, <)$ is a partial order and $F, G \subseteq X$, we write $F \sqsubseteq G$ if F is an initial segment of G, i.e., $F \subseteq G$ and for any $y \in F$ and each $x \in G$, if $x \leq y$, then $x \in F$. Given two elements of X, say x and y, x and y are said to be comparable if either $x \leq y$ or $y < x$. If there is $z \in X$ such that $z \leq x$ and $z \leq y$, we say that x and y are compatible. Whenever x and y are not comparable we say that they are incomparable, and whenever they are not compatible then they are incompatible. If A is a subset of X and we write $x < y \in A$, we will usually mean that both x and y are members of A. Whenever this is not the case and it is not clear from context, we clarify it. By (x, y) and $[x, y]$ we denote the sets $\{z \in X : x < z < y\}$ and $\{z \in X : x \leq z \leq y\}$ respectively. $(x, y]$ and $[x, y)$ are defined in an analogous way. We say that $A \subseteq X$ is an interval if $(x, y) \subseteq A$ whenever $x, y \in A$. We say that A is a chain if every two elements of A are comparable. If all the elements of A are pairwise incomparable we call A a *pie*. A similar notion which is more frequently used is the one of antichain. A is said to be an *antichain* if every two elements of A are incompatible. A is cofinal if every element of X is below some element of A. We define the cofinality of X as $cof(X) = \min(|A| : A \subseteq X \textit{ is cofinal})$. X is said to be *well-founded* if any non-empty subset of X has a minimal element. In this case, there is a unique function $rank : X \longrightarrow Ord$ satisfying:

$$rank(x) = \sup(\, rank(y) + 1 \,:\, y < x\,)$$

Note that this function is order preserving and its image is an ordinal. For each ordinal α, we define the *level* α *of* X as $X_\alpha = \{x \in X : rank(x) = \alpha\}$ and the *height of* X as $Ht(X) = \min(\,\alpha \in Ord : X_\alpha = \emptyset\,) = rank[X]$. X is said to be a *well-order* if any non-empty subset of X has a minimum. In this case, there is a unique ordinal β and a unique order preserving function $\phi : \alpha \longrightarrow X$ called *the enumeration of* X. We call β the order type of X and denote it as $ot(X)$. In the previous situation we identify X with ϕ. In this way, $X(\alpha)$ denotes $\phi(\alpha)$ and $X^{-1}(x)$ denotes $\phi^{-1}(x)$ whenever $\alpha \in \beta$ and $x \in X$. In the same way, $X[S]$ denotes $\phi[S]$ and $X^{-1}[Y]$ denotes $\phi^{-1}[Y]$ whenever $S \subseteq \alpha$ and $Y \subseteq X$.

2. Construction Schemes

We will start this section by defining construction schemes following the lines of [77]. For convenience of the reader, we will then sketch the proofs of some basic properties of schemes (all of which already appear in [42] and [77]).

Definition 2.1 (Type). *We call a sequence* $\tau = \langle m_k, n_{k+1}, r_{k+1}\rangle_{k\in\omega} \subseteq \omega^3$ *a type if:*

(a) $m_0 = 1$,
(b) $\forall 0 < k \in \omega(\, n_k \geq 2\,)$,
(c) $\forall k \in \omega(\, m_k > r_{k+1}\,)$,
(d) $\forall k \in \omega(\, m_{k+1} = r_{k+1} + (m_k - r_{k+1})\, n_{k+1}\,)$.

We say that τ *is good if:*

(e) $\forall r \in \omega\, \exists^\infty k \in \omega(\, r_k = r\,)$,

Additionally, we say that a partition of ω*, namely* $\mathcal{P}$*, is compatible with* τ *if:*

(e') $\forall P \in \mathcal{P}\, \forall r \in \omega\, \exists^\infty k \in P(\, r_k = r\,)$.

It is easy to see that points (b) and (d) of the previous definition imply that the sequence of m_k's is increasing.

If X is an infinite set and $\mathcal{F} \subseteq [X]^{<\omega}$, then $(\mathcal{F}, \subsetneq)$ is well-founded. Hence, for each ordinal α we can define $\mathcal{F}_\alpha$ as in the preliminaries . In this particular case we have that $\mathcal{F} = \bigcup_{k\in\omega} \mathcal{F}_k$. Furthermore, if $\mathcal{F}$ is cofinal in $[X]^{<\omega}$ then $\mathcal{F}_k \neq \emptyset$ whenever $k \in \omega$.

Definition 2.2 (Construction scheme)**.** *Let $\tau = \langle m_k, n_{k+1}, r_{k+1}\rangle_{k\in\omega}$ be a type, and X be a set of ordinals. We say that $\mathcal{F} \subseteq FIN(X)$ is a construction scheme (or simply a scheme) over X of type τ if:*

(a) $\mathcal{F}$ *is cofinal in* $FIN(X)$,
(b) $\forall k \in \omega\ \forall F \in \mathcal{F}_k\big(\ |F| = m_k\ \big)$,
(c) $\forall k \in \omega\ \forall F, E \in \mathcal{F}_k\big(\ E \cap F \sqsubseteq E, F\ \big)$,
(d) $\forall k \in \omega\ \forall F \in \mathcal{F}_{k+1}\ \exists F_0, \ldots, F_{n_{k+1}-1} \in \mathcal{F}_k$ *such that*
$$F = \bigcup_{i<n_{k+1}} F_i.$$
Moreover, $\langle F_i\rangle_{i<n_{k+1}}$ forms a Δ-system with root $R(F)$ such that $|R(F)| = r_{k+1}$ and
$$R(F) < F_0\backslash R(F) < \cdots < F_{n_{k+1}-1}\backslash R(F).$$

If $n \in \omega$ and $n_{k+1} = n$ for each $k \in \omega$, we will call $\mathcal{F}$ an n-construction scheme (or simply an n-scheme).

The following theorem was proved by the third author in [77]. The interested reader may find another proof in [42].

Theorem 2.3 ([77])**.** *For every good type τ there is a construction scheme over ω_1 of that type.*

Simplified morasses as defined in [87] and construction schemes are closely related. In fact, 2-construction schemes are the objects which we also know as $(\omega, 1)$-gap simplified morasses. Readers who want to know more about morasses and their applications may look at [10], [15], [32], [63], [68], [87], [88], and [89].

For the rest of this section, unless otherwise stated we fix a type $\langle m_k, n_k, r_{k+1}\rangle_{k\in\omega_1}$. The next lemma is an easy consequence of the condition (d) in Definition 2.2.

Lemma 2.4. *Let $\mathcal{F}$ be a construction scheme over X, $l \in \omega$ and $F \in \mathcal{F}_l$. For each $k \leq l$, it happens that $F = \bigcup\{H \in \mathcal{F}_k\ :\ H \subseteq F\}$.*

There is a version of condition (c) in Definition 2.2 in the case where the sets have different rank.

Lemma 2.5. *Let $\mathcal{F}$ be a construcion scheme over X and let $k \leq l \in \omega$. If $G \in \mathcal{F}_k$ and $F \in \mathcal{F}_l$ then $G \cap F \sqsubseteq G$.*

Proof. As a consequence of Lemma 2.4 we know that
$$\begin{aligned} G \cap F &= G \cap \big(\bigcup\{\ H \in \mathcal{F}_k\ :\ H \subseteq F\}\big) \\ &= \bigcup_{H\in\mathcal{F}_k\cap\mathscr{P}(F)} G \cap H. \end{aligned}$$

As each of the sets forming the union of the last equality is an initial segment of G by condition (c) of Definition 2.2, such union is also an initial segment of G. $\square$

Remark 2.6. *In general it is not true that if $k < l$, $G \in \mathcal{F}_k$ and $F \in \mathcal{F}_l$ then $G \cap F$ is an initial segment of F. An example of this ocurrs when G is the second piece of the canonical decomposition of F, namely F_1.*

Proposition 2.7. *Let $\mathcal{F}$ be a construction scheme over X and let $k \in \omega$ be such that $m_k \leq |X|$. Then $X = \bigcup \mathcal{F}_k$.*

Proof. Let $\alpha \in X$ and let Y be a subset of X of cardinality m_k. Due to condition (a) of Definition 2.2 there is $l \in \omega$ and $F \in \mathcal{F}_l$ for which $\{\alpha\} \cup Y \subseteq F$. Note that $l \geq k$ as $|Y| = m_k$. Therefore, by Lemma 2.4 we conclude that $F = \bigcup\{G \in \mathcal{F}_k : G \subseteq F\}$. In particular $\alpha \in G$ for some $G \in \mathcal{F}_k$ with $G \subseteq F$. This finishes the proof. $\square$

Remark 2.8. *By applying the previous proposition to the case where $k = 0$ we conclude that $[X]^1 = \mathcal{F}_0$.*

3. Construction schemes, ordinal metrics and related functions

In this section we study the connection between construction schemes and ordinal metrics (introduced by the third author in [82]). Together with the walks on ordinals, they have proven to be invaluable tools in the study of ω_1. The book [74] contains a considerable amount of information about these two objects. Before starting with the definitions it is worth pointing out that Charles Morgan presented in [55] a detailed analysis of the relation between ordinal metrics and gap-1 morasses.

Definition 3.1 (Closure operation)**.** *Let $\rho : X^2 \longrightarrow \omega$ be an arbitrary function. For each $F \in FIN(X)$ and $k \in \omega$ we define the k-*closure *of F as:*

$$(F)_k = \{\, \alpha \in X \,:\, \exists \beta \in F \backslash \alpha \, (\rho(\alpha, \beta) \leq k) \,\}$$
$$(F)_k^- = (F)_k \cap \max(F) = (F)_k \backslash \{\max(F)\}$$

The diameter *of F is defined as*

$$\rho^F = \max(\, \rho(\alpha, \beta) \,:\, \alpha, \beta \in F \,).$$

*F is said to be k-*closed *whenever $F = (F)_k$. Moreover, if F is ρ^F-closed we just say that F is* closed*. For $\alpha \in X$, we will write $(\alpha)_k$ and $(\alpha)_k^-$ instead of $(\{\alpha\})_k$ and $(\{\alpha\})_k^-$ respectively. Finally, F is a maximally closed set (with respect to ρ) if F is closed and there is no closed set G such that $\rho^F = \rho^G$ and $F \subsetneq G$.*

Remark 3.2. *Both the k-closure and the diameter are monotone operations. That is, if $F \subseteq G$ then $(F)_k \subseteq (G)_k$ and $\rho^F \leq \rho^G$.*

Following the convention established in the preliminaries, if $(\alpha)_k = \{\beta_0, \ldots, \beta_n\}$ with $\beta_0 < \cdots < \beta_n$, we write $(\alpha)_k(i) = \beta_i$ for each $i \leq n$.

Definition 3.3 (k-cardinality)**.** *Let ρ be as in the previous definition. Given $\alpha \in X$ and $k \in \omega$ we define the k-*cardinality *of α as*

$$\|\alpha\|_k = |(\alpha)_k^-|.$$

Remark 3.4. *Given $\rho : X^2 \longrightarrow \omega$ and $\beta \in X$, we have that $(\beta)_k = \{\alpha \leq \beta \mid \rho(\alpha, \beta) \leq k\}$.*

We proceed to define ordinal metrics following the same approach (although slightly different notation) as in [74]. In practice, we will only be interested in ordinal metrics whose domain is ω_1^2. However, such metrics are constructed via recursion. For that reason, it is useful to present the definition in the following generality.

Definition 3.5 (Ordinal metric)**.** [3] *We say that* $\rho : X^2 \longrightarrow \omega$ *is an* ordinal metric *(over* X*) if:*

(a) $\forall \alpha, \beta \in X\ (\ \rho(\alpha,\beta) = 0$ *if and only if* $\alpha = \beta\ )$,
(b) $\forall \alpha, \beta \in X\ (\ \rho(\alpha,\beta) = \rho(\beta,\alpha)\ )$,
(c) $\forall \alpha, \beta, \gamma \in X\ ($*if* $\alpha \leq \beta, \gamma,$ *then* $\rho(\alpha,\beta) \leq \max(\rho(\alpha,\gamma), \rho(\beta,\gamma))\ )$,
(d) $\forall \beta \in X\ \forall k \in \omega\ (\ \|\beta\|_k < \omega\ )$.

As the reader may note, the previous definition reassembles that of an ultrametric[4]. However, note that one of the instances of the triangle inequality is missing. In this sense, we can interpret $(\beta)_k$ as the ball with radius k centered at β intersected with $\beta + 1$. In the theory of metric spaces, we usually want to learn about the nature of points in a given space. To do that, we study the behaviour of their basic neighbourhoods as they become smaller. Ordinal metrics tend to work in the opposite way. Here, we want to construct a structure whose elements are parametrized with the points of our space. In order to do that, we make suitable approximations of our structure by analyzing the interaction between β and other points in $(\beta)_k$ as k grows.
The following propositions are direct consequences of the definitions.

Proposition 3.6. *Let* $\rho : X^2 \longrightarrow \omega$ *be and ordinal metric,* $F \in FIN(X)$ *and* $k \geq \rho^F$*. Then:*

(1) $\rho^F = \max(\ \rho(\alpha, \max(F))\ :\ \alpha \in F)$.
(2) $(F)_k = \{\ \alpha \leq \max(F)\ :\ \rho(\alpha, \max(F)) \leq k\ \} = (\max(F))_k$.
(3) *If* $\beta \in F$*, then* $(F)_k \cap (\beta + 1) = (F \cap (\beta + 1))_k$.
(4) $(F)_k$ *is* k*-closed and* $\rho^{(F)_k} \leq k$*, so in particular* $(F)_k$ *is closed. Furthermore, if* $k = \rho^F$*, then* $\rho^{(F)_k} = k$.
(5) F *is closed if and only if* $F = (\max(F))_{\rho^F}$.
(6) *If* $G \in FIN(X)$ *is* k*-closed, then* $F \cap G \sqsubseteq F$.

Proposition 3.7. *Let* $\rho : X^2 \longrightarrow \omega$ *be an ordinal metric and let* $\mathcal{A} \subseteq FIN(X)$ *be a family of closed sets of diameter* k*. If* $F \in FIN(X)$ *is such that* $\rho^F \leq k$ *and* $F \subseteq \bigcup \mathcal{A}$ *then there is* $G \in \mathcal{A}$ *for which* $F \subseteq G$.

Suppose that $\mathcal{F}$ is a construction scheme. Given $\alpha, \beta \in X$, there is $F \in \mathcal{F}$ which contains both α and β. We can define an ordinal metric over X by looking at the minimum k for which such F exists.

Definition 3.8 (The ρ-function)**.** *Let* $\mathcal{F}$ *be a construction scheme over* X*. We define* $\rho_{\mathcal{F}} : X^2 \longrightarrow \omega$ *as:*

$$\rho_{\mathcal{F}}(\alpha, \beta) = \min(\ k \in \omega\ :\ \exists F \in \mathcal{F}_k(\ \{\alpha, \beta\} \subseteq F\ )\ ).$$

If $\mathcal{F}$ *is clear from context, we will write* $\rho_{\mathcal{F}}$ *simply as* ρ.

[3] Some authors prefer to call ordinal metrics "T-functions" or "ρ-functions".
[4] A metric d over a space X is an ultrametric if $d(x,y) \leq \max(d(x,z), d(z,y))$ for all $x, y, z \in X$.

In the following lemma we characterize the k-closures with respect to the ordinal metric which we defined above. Note that in the proof, we are not assuming that ρ is indeed an ordinal metric.

Lemma 3.9. *Let $\mathcal{F}$ be a construction scheme over X. If $\beta \in X$ and $k \in \omega$, then*

$$(\beta)_k = F \cap (\beta + 1)$$

for each $F \in \mathcal{F}_k$ with $\beta \in F$.

Proof. By Definition 3.8 it is clear that $F \cap (\beta+1) \subseteq (\beta)_k$. It only remains to prove the other inclusion. For this, let $\alpha \in (\beta)_k$ and $G \in \mathcal{F}_{\rho(\alpha,\beta)}$ be such that $\{\alpha, \beta\} \subseteq G$. By the definition of the k-closure we know that $\rho(\alpha, \beta) \leq k$. Hence, $G \cap F \sqsubseteq G$ due to Lemma 2.5. As $\beta \in G \cap F$ and $\alpha \leq \beta$ it must happen that $\alpha \in G \cap F$. In particular $\alpha \in F$, so we are done. $\square$

For the sake of completeness, we will now prove that ρ is indeed an ordinal metric.

Proposition 3.10. *Let $\mathcal{F}$ be a construction scheme over X. Then $\rho = \rho_{\mathcal{F}}$ is an ordinal metric.*

Proof. The condition (b) of Definition 3.5 is trivially satisfied, the condition (a) is a consequence of the Remark 2.8 and the condition (d) follows from Lemma 3.9. Thus, we only need to prove that the condition (c) of such definition also holds. For this, let $\alpha, \beta, \gamma \in X$ be such that $\alpha \leq \beta, \gamma$. Consider $F \in \mathcal{F}_{\rho(\alpha,\gamma)}$ and $G \in \mathcal{F}_{\rho(\beta,\gamma)}$ for which $\{\alpha, \gamma\} \subseteq F$ and $\{\beta, \gamma\} \subseteq G$. We need to consider two cases.

<u>Case 1</u>: If $\rho(\alpha, \gamma) \leq \rho(\beta, \gamma)$.

Proof of case. Here $F \cap G \sqsubseteq F$ by Lemma 2.5. As $\gamma \in F \cap G$ and $\alpha \leq \gamma$ we have that $\alpha \in F \cap G$. In particular $\alpha \in G$ so $\{\alpha, \beta\} \subseteq G$. Therefore $\rho(\alpha, \beta) \leq \rho(\beta, \gamma)$. $\square$

<u>Case 2</u>: If $\rho(\beta, \gamma) \leq \rho(\alpha, \gamma)$.

Proof of case. In this case $F \cap G \sqsubseteq G$. If $\beta \leq \gamma$ then $\beta \in F \cap G$ because $\gamma \in F \cap G$. Therefore $\{\alpha, \beta\} \subseteq G$ which means $\rho(\alpha, \beta) \leq \rho(\beta, \gamma)$. On the other hand, if $\beta \geq \gamma$ we can use Proposition 2.7 to pick $F' \in \mathcal{F}_{\rho(\alpha,\gamma)}$ for which $\beta \in F'$. Since $F' \cap (\beta + 1) = (\beta)_{\rho(\alpha,\gamma)}$ by Lemma 3.9 then $\gamma \in F'$. Again, by the same lemma, $F' \cap (\gamma + 1) = (\gamma)_{\rho(\alpha,\gamma)}$ which means that $\alpha \in F'$. Consequently $\{\alpha, \beta\} \subseteq F'$ so $\rho(\alpha, \beta) \leq \rho(\alpha, \gamma)$. $\square$

$\square$

Together with Lemma 3.9, the following three results yield the most basic properties of $\rho_{\mathcal{F}}$.

Proposition 3.11. *Let $\mathcal{F}$ be a construction scheme over X, $\rho = \rho_{\mathcal{F}}$ and let $F \in \mathcal{F}_k$ for some $k > 0$. Consider $\langle F_i \rangle_{i<n_k}$ the canonical decomposition of F as described in point (d) of Definition 2.2. If $\alpha \in F_i \backslash R(F)$ and $\beta \in F_j \backslash R(F)$ for $i < j < n_k$ then $\rho(\alpha, \beta) = k$.*

Proof. First observe that $\rho(\alpha, \beta) \leq k$ as testified by F. Now, $(\beta)_{k-1} = F_j \cap (\beta + 1)$ due to the Lemma 3.9. In this way, $\alpha \notin (\beta)_{k-1}$, which means that $\rho(\alpha, \beta) > k - 1$. This finishes the proof. $\square$

Lemma 3.12. *Let $\mathcal{F}$ be a construction scheme over X and $\rho = \rho_{\mathcal{F}}$. For every $k \in \omega$ and $F \in \mathcal{F}_k$, we have that $\rho^F = k$.*

Proof. For $k=0$ the result follows from Remark 2.8, so we will only consider the case where $k>0$. Let $F\in\mathcal{F}_k$. By Definition 3.8 it should be clear that $\rho^F\leq k$. To prove the other inequality let $\langle F_i\rangle_{i<n_k}$ be the canonical decomposition of F as described in point (d) of Definition 2.2. Now, consider $\alpha\in F_0\backslash R(F)$ and $\beta\in F_1\backslash R(F)$. By Proposition 3.11, $\rho(\alpha,\beta)=k$. Hence $k\leq\rho(\alpha,\beta)\leq\rho^F$. $\square$

For each construction scheme $\mathcal{F}$, the maximally closed sets with respect to $\rho_{\mathcal{F}}$ are exactly the members of $\mathcal{F}$. We leave the proof of the two following facts to the reader.

Lemma 3.13. *Let $\mathcal{F}$ be a construction scheme over X and $\rho=\rho_{\mathcal{F}}$. Each element of $\mathcal{F}$ is maximally closed with respect to ρ.*

Proposition 3.14. *Let $\mathcal{F}$ be a construction scheme over X and $\rho=\rho_{\mathcal{F}}$. Then*

$$\mathcal{F}_k=\{\,H\in FIN(X)\,:\,H \text{ is maximally closed and } \rho^H=k\,\}$$

for each $k\in\omega$.

It turns out that construction schemes are in a one-to-one correspondence with class of ordinal metrics. To state this precisely, we need the following definitions.

Definition 3.15. *Let $\rho:X^2\longrightarrow\omega$ be an ordinal metric and let $F,G\in FIN(X)$. We say that F and G are ρ-isomorphic if $|F|=|G|$ and for all $i,j<|F|$,*

$$\rho(F(i),F(j))=\rho(G(i),G(j)).$$

Equivalently, if $h:F\longrightarrow G$ is the increasing bijection[5] then

$$\rho(\alpha,\beta)=\rho(h(\alpha),h(\beta))$$

for all $\alpha,\beta\in F$.

Definition 3.16. *Let $\rho:X^2\longrightarrow\omega$ be an ordinal metric. We will say that ρ is:*

(1) locally finite *if $\sup\{\,|F|\;:\;F\in FIN(X),\, F$ is closed and $\rho^F\leq k\}<\omega$ for every $k\in\omega$.*

(2) homogeneous *if whenever $F,G\in FIN(X)$ are maximally closed sets such that $\rho^F=\rho^G$, then F and G are ρ-isomorphic.*

(3) regular *if for each $0<k\in\omega$ and each maximally closed $F\in FIN(X)$ with $\rho^F=k$ there are $1\leq j_F\in\omega$ and $F_0,\dots,F_{j_F}\in FIN(X)$such that*

- *$F=\bigcup\limits_{i<j_F}F_i$,*
- *for each $i<j_F$, F_i is a maximally closed set with $\rho^{F_i}=k-1$,*
- *$\langle F_i\rangle_{i<j_F}$ forms a Δ-system with root $R(F)$ such that*

$$R(F)<F_0\backslash R(F)<\cdots<F_{j_F-1}\backslash R(F).$$

If an homogenuous ordinal metric is locally finite and regular, then we can prove that a great variety of pairs of (not necessarily maximally) closed sets are ρ-isomorphic. As we do not want to stray from the main topic, we will leave the proofs to the reader.

Lemma 3.17. *Suppose that $\rho:X^2\longrightarrow\omega$ is a locally finite and homogeneous ordinal metric. If $F,G\in FIN(X)$ are two closed sets (not necessarily maximal) with $|F|=|G|$ and $\rho^F=\rho^G$ then F and G are ρ-isomorphic.*

[5]Note that if $|F|=|G|$ then h is defined by the formula $h(F(i))=G(i)$.

Corollary 3.18. *Let $\rho : X^2 \longrightarrow \omega$ be a regular, locally finite and homogeneous ordinal metric. If $\alpha, \beta \in X$ and $k \in \omega$ is such that $\|\alpha\|_k = \|\beta\|_k$ then $(\alpha)_k$ and $(\beta)_k$ are ρ-isomorphic.*

The next three results yield a complete characterization of construction schemes in terms of ordinal metrics.

Proposition 3.19. *Let $\rho : X^2 \longrightarrow \omega$ be a regular, locally finite and homogeneous ordinal metric. The set*

$$\mathcal{F}^\rho = \{ F \in FIN(X) : F \text{ is maximally closed} \}$$

is a construction scheme over X.

Theorem 3.20. *Let $\mathcal{F}$ be a construction scheme over X. Then $\rho = \rho_{\mathcal{F}}$ is locally finite, regular and homogeneous.*

Corollary 3.21. *If $\mathcal{F}$ is a construction scheme over X and $\rho : X^2 \longrightarrow \omega$ is a locally finite, homogeneous and regular ordinal metric, then:*

- $\mathcal{F}^{\rho_{\mathcal{F}}} = \mathcal{F}$,
- $\rho_{\mathcal{F}^\rho} = \rho$.

Before advancing any further, we think it is convenient to remark one more time that ordinal metrics (over ω_1) should not be treated as metrics. In fact missing triangle inequality from Definition 3.5 is never satisfied.

Proposition 3.22. *No ordinal metric over ω_1 is a metric.*

Proof. Suppose ρ is as previously mentioned and let M be a countable elementary submodel of $H(\omega_2)$ having ρ as an element. We will show that ρ does not satisfy the triangle inequality of metric spaces. Take $\gamma \in \omega_1 \backslash M$, consider an arbitrary $\alpha \in M \cap \omega_1$ and define k as $\rho(\alpha, \gamma)$. Using elementarity we can find $\beta \in M \cap \omega_1$ above $\max((\gamma)_{2k} \cap M)$ for which $k = \rho(\alpha, \beta)$. By definition of the k-closure, we have that $\rho(\beta, \gamma) > 2k = \rho(\alpha, \beta) + \rho(\alpha, \gamma)$. This finishes the proof. □

3.1. **More functions associated to a construction scheme.** Apart from the ordinal metric associated to a construction scheme, there are two important functions that we need to analyze before we enter to the world of applications. The first one being the Δ function and the second one being the Ξ function. For the rest of this subsection we fix a set of ordinals X and a construction scheme $\mathcal{F}$.

The Δ function just measures the least moment in which the k-cardinality of two ordinals differs.

Definition 3.23 (The Δ function)**.** *We define $\Delta : X^2 \longrightarrow \omega + 1$ as:*

$$\Delta(\alpha, \beta) = \begin{cases} \min(k \in \omega \, : \, \|\alpha\|_k \neq \|\beta\|_k) & \text{if } \alpha \neq \beta \\ \omega & \text{if } \alpha = \beta \end{cases}$$

Remark 3.24. *Δ is well defined since $\|\alpha\|_{\rho(\alpha,\beta)} \neq \|\beta\|_{\rho(\alpha,\beta)}$ whenever $\alpha \neq \beta$. Moreover, $\Delta(\alpha, \beta) \leq \rho(\alpha, \beta)$.*

Lemma 3.25. *Let $\alpha, \beta \in X$ be distinct ordinals and $k \in \omega$. If $\|\alpha\|_k = \|\beta\|_k$ then $k < \Delta(\alpha, \beta)$.*

Proof. By Corollary 3.18 $(\alpha)_k$ is ρ-isomorphic to $(\beta)_k$. Let $h : (\alpha)_k \longrightarrow (\beta)_k$ be the increasing bijection. Note that $h(\alpha) = \beta$. Now consider an arbitrary $i \leq k$. Then $(\alpha)_i \subseteq (\alpha)_k$. In this way, $h[(\alpha)_i] = (h(\alpha))_i = (\beta)_i$. This means that $\|\alpha\|_i = \|\beta\|_i$. Therefore $k < \Delta(\alpha, \beta)$. $\square$

Recall that for any two functions f, g from ω to ω, we can define $\Delta(f, g) = \min(k \in \omega : f(k) \neq g(k))$. It should be clear that $\Delta(\|\alpha\|_-, \|\beta\|_-) = \Delta(\alpha, \beta)$ for any $\alpha, \beta \in X$. Hence, we have the following lemma.

Lemma 3.26. *Let $\alpha, \beta, \delta \in X$ be distinct ordinals such that $\Delta(\alpha, \beta) < \Delta(\beta, \delta)$. Then $\Delta(\alpha, \delta) = \Delta(\alpha, \beta)$.*

Lemma 3.27. *Let $\alpha, \beta \in X$ be distinct ordinals, $k < \Delta(\alpha, \beta)$ and $h : (\alpha)_k \longrightarrow (\beta)_k$ be the increasing bijection.*

If $\delta \in (\alpha)_k$ then:

(*a*) $\Delta(\delta, h(\delta)) \geq \Delta(\alpha, \beta)$,
(*b*) $\rho(\alpha, \beta) \geq \rho(\delta, h(\delta))$.

In other words, if $i < \|\alpha\|_k$ then:

(*a*) $\Delta((\alpha)_k(i), (\beta)_k(i)) \geq \Delta(\alpha, \beta)$,
(*b*) $\rho(\alpha, \beta) \geq \rho((\alpha)_k(i), (\beta)_k(i))$.

Proof. Proof of (*a*). Let $l = \Delta(\alpha, \beta) - 1$ and $h' : (\alpha)_l \longrightarrow (\beta)_l$ be the increasing bijection. As $k < \Delta(\alpha, \beta)$ then $(\alpha)_k \subseteq (\alpha)_l$ and $h'[(\alpha)_k] = (\beta)_k$. Hence $h'|_{(\alpha)_k} = h$ so $h'(\delta) = h(\delta)$. Therefore $h'[(\delta)_l] = (h(\delta))_l$. In this way, $\|\delta\|_l = \|h(\delta)\|_l$. Thus, by Lemma 3.25 we conclude that $\Delta(\alpha, \beta) \leq \Delta(\delta, \phi(\delta))$. $\square$

Proof of (*b*). Let $F \in \mathcal{F}_{\rho(\alpha,\beta)}$ be such that $\alpha, \beta \in F$. Since $k < \Delta(\alpha, \beta) \leq \rho(\alpha, \beta)$ then $(\alpha)_k \cup (\beta)_k \subseteq F$. In particular $\delta, h(\delta) \in F$ so $\rho(\alpha, \beta) \geq \rho(\delta, h(\delta))$. $\square$

$\square$

Remark 3.28. *If α and β are so that $\|\alpha\|_k = \|\beta\|_k$ and $h : (\alpha)_k \longrightarrow (\beta)_k$ is the increasing bijection, then the following happens:*

- $h(\alpha) = h(\beta)$,
- $(\alpha)_k \cap (\beta)_k \sqsubseteq (\alpha)_k, (\beta)_k$,
- $h|_{(\alpha)_k \cap (\beta)_k}$ *is the identity function,*
- $h[(\alpha)_k \backslash (\beta)_k] = (\beta)_k \backslash (\alpha)_k$.

Lemma 3.29. *Let $\alpha, \beta \in X$ be distinct ordinals and $k < \Delta(\alpha, \beta)$. If $h : (\alpha)_k \longrightarrow (\beta)_k$ is the increasing bijection and $\delta, \gamma \in (\alpha)_k$ are such that $\delta \leq \gamma$ and $h(\delta) \neq \delta$ then the following happens:*

(*a*) $h(\gamma) \neq \gamma$,
(*b*) $\rho(\alpha, \beta) \geq \rho(\gamma, h(\gamma)) \geq \rho(\delta, h(\delta)) \geq \Delta(\delta, h(\delta)) \geq \Delta(\gamma, h(\gamma)) \geq \Delta(\alpha, \beta)$.

Proof. Remember that $C = (\alpha)_k \cap (\beta)_k$ is an initial segment of both $(\alpha)_k$ and $(\beta)_k$. In this way, it is easy to see that $h(\xi) = \xi$ if and only if $\xi \in C$. Note that since $\delta \notin C$ and $\gamma > \delta$, then $\gamma \notin C$. Therefore $h(\gamma) \neq \gamma$. This proves the point (a).

The inequalities from (b) follow directly from Lemma 3.27 and the fact that $h|_{(\gamma)_k}$ is the increasing bijection from $(\gamma)_k$ to $(h(\gamma))_k$.

$\square$

Definition 3.30 (The Ξ function)**.** *Let $\alpha \in X$. $\Xi_\alpha : \omega \longrightarrow \omega \cup \{-1\}$ is the function defined as:*

$$\Xi_\alpha(k) = \begin{cases} 0 & \text{if } k = 0 \\ -1 & \text{if } k > 0 \text{ and } \|\alpha\|_k \leq r_k \\ \frac{\|\alpha\|_k - \|\alpha\|_{k-1}}{m_{k-1} - r_k} & \text{otherwise} \end{cases}$$

It is not hard to check that if $k \in \omega \backslash 1$ and $F \in \mathcal{F}_k$ is such that $\alpha \in F$, then:

$$\Xi_\alpha(k) = \begin{cases} -1 & \text{if } \alpha \in R(F) \\ i & \text{if } \alpha \in F_i \backslash R(F) \end{cases}$$

Lemma 3.31. *Let $\alpha < \beta \in X$ and $k \in \omega \backslash 1$. Then:*

(a) *If $k < \Delta(\alpha, \beta)$, then $\Xi_\alpha(k) = \Xi_\beta(k)$.*
(b) *If $k = \rho(\alpha, \beta)$, then $0 \leq \Xi_\alpha(k) < \Xi_\beta(k)$.*
(c) *If $k > \rho(\alpha, \beta)$, then either $\Xi_\alpha(k) = -1$ or $\Xi_\alpha(k) = \Xi_\beta(k)$.*
(d) *If $k = \Delta(\alpha, \beta)$ then $0 \leq \Xi_\alpha(k) \neq \Xi_\beta(k) \geq 0$.*

Proof. (a) is direct from the definition of $\Delta(\alpha, \beta)$ so we will only prove the remaining points.

Proof of (*b*). Fix $F \in \mathcal{F}_{\rho(\alpha,\beta)}$ for which $\alpha, \beta \in F$. We know there are $i, j < n_k$ such that $\alpha \in F_i$ and $\beta \in F_j$. Since $F_i, F_j \in \mathcal{F}_{\rho(\alpha,\beta)-1}$, minimality of ρ implies that $\alpha \notin F_j$ and $\beta \notin F_i$. In particular, this means $i \neq j$ and $\alpha, \beta \notin R(F)$. Thus, by Definition 3.30 we have that $\Xi_\alpha(k) = i$ and $\Xi_\beta(k) = j$. Moreover, since $\alpha < \beta$ then $i < j$. □

Proof of (*c*). For this, suppose that $\Xi_\alpha(k) \neq -1$ and let $F \in \mathcal{F}_k$ be such that $\alpha, \beta \in F$. Since $\alpha < \beta$, we also have $\Xi_\beta(k) \neq -1$. Thus, there is $j < n_k$ for which $\beta \in F_j \backslash R(F)$. Recall that $F_j \in \mathcal{F}_{k-1}$. In this way, $\alpha \in (\beta)_{k-1} = (\beta + 1) \cap F_j$. As $\alpha \notin R(F)$, this means $\Xi_\alpha(k) = j = \Xi_\beta(k)$. So we are done. □

Proof of (*d*). Let $F, G \in \mathcal{F}_k$ be such that $\alpha \in F$ and $\beta \in G$. Also, let $h : F \longrightarrow G$ be the increasing bijection. Then $h[(\alpha)_i] = (h(\alpha))_i$ for any $i \leq k$. In this way,

$$k = \Delta(\alpha, \beta) = \Delta(h(\alpha), \beta) \leq \rho(h(\alpha), \beta) \leq k.$$

That is, $k = \rho(h(\alpha), \beta)$. So by the part (b) of this lemma we conclude that $\Xi_{h(\alpha)}(k)$ and $\Xi_\beta(k)$ are both distinct and greater or equal to 0. To finish just note that $\Xi_\alpha(k) = \Xi_{h(\alpha)}(k)$. □

□

As an easy consequence of the previous lemma we have the following corollary.

Corollary 3.32. *Suppose that $X = \omega_1$ and let $\alpha \in X$. Then:*

(1) $\Xi_\alpha(k) = -1$ *for infinitely many $k \in \omega$.*
(2) *If α is infinite, then $\Xi_\alpha(k) \geq 1$ for infinitely many $k \in \omega$.*
(3) *If τ (the type of $\mathcal{F}$) is a good type, then $\Xi_\alpha(k) \geq 0$ for infinitely many $k \in \omega$.*

Proof. In order to prove (3) just observe that since τ is a good type then $r_k = 0$ for infinitely many k's. For any such k it is necessarily true that $\Xi_\alpha(k) \geq 0$. We now prove the remaining points.

Proof of (1). Let $l \in \omega$. We will find $k > l$ for which $\Xi_\alpha(k) = -1$. Since $X = \omega_1$ there is an uncountable $S \subseteq \omega_1 \backslash \alpha$ and $s \in \omega$ such that $\rho(\alpha, \beta) = \rho(\alpha, \delta) = s$ for any $\beta, \delta \in S$. It is easy to see that there are $\beta < \delta \in S$ for which $\rho(\beta, \delta) > \max(l, s)$. Let $k = \rho(\beta, \delta)$. By the point (b) in Lemma 3.31 we know that $0 \leq \Xi_\beta(k) < \Xi_\delta(k)$. If $\Xi_\alpha(k) \neq -1$ then we would have that $\Xi_\beta(k) = \Xi_\alpha(k) = \Xi_\delta(k)$ due to part (c) of that same lemma, which is impossible. Hence, we are done. $\square$

Proof of (2). Fix $l \in \omega$. We shall find $k > l$ for which $\Xi_\alpha(k) \geq 1$. Since $(\alpha)_l$ is finite and α is infinite there is $\xi \in \alpha \backslash (\alpha)_l$. By the definition of the l-closure we have that $k = \rho(\xi, \alpha) > l$. Then $\Xi_\alpha(k) \geq 1$ do to the point (b) of Lemma 3.31. $\square$

$\square$

Lemma 3.33. *Let $\alpha, \beta \in X$ be distinct ordinals, $k < \Delta(\alpha, \beta)$ and $h : (\alpha)_k \longrightarrow (\beta)_k$ be the increasing bijection. If $\gamma \in (\alpha)_k$ is such that $\gamma \neq h(\gamma)$ then the following are equivalent:*

(a) $\Delta(\gamma, h(\gamma)) > \Delta(\alpha, \beta)$.
(b) $\Xi_\gamma(\Delta(\alpha, \beta)) = -1$.

Furthermore, if $\Xi_\gamma(\Delta(\alpha, \beta) \geq 0$ then $\Xi_\gamma(\Delta(\alpha, \beta)) = \Xi_\alpha(\Delta(\alpha, \beta))$ and $\Xi_{h(\gamma)}(\Delta(\alpha, \beta)) = \Xi_\beta(\Delta(\alpha, \beta))$.

Proof. Proof of (a) $\Rightarrow$ (b). We argue by contradiction. Suppose that $\Xi_\gamma(\Delta(\alpha, \beta)) \geq 0$. Since $\rho(\alpha, \gamma) \leq k < \Delta(\alpha, \beta)$ then $\Xi_\gamma(\Delta(\alpha, \beta)) = \Xi_\alpha(\Delta(\alpha, \beta)) \geq 0$ by the part (c) of Lemma 3.31. Now, since we are assuming that $\Delta(\gamma, h(\gamma)) > \Delta(\alpha, \beta)$, then $0 \leq \Xi_\gamma(\Delta(\alpha, \beta)) = \Xi_{h(\gamma)}(\Delta(\alpha, \beta))$ due to the point (a) of Lemma 3.31. Thus, we can argue in the same way as before to conclude that $\Xi_{h(\gamma)}(\Delta(\alpha, \beta)) = \Xi_\beta(\Delta(\alpha, \beta))$. Therefore, according the part (d) of the same lemma,

$$\Xi_\gamma(\Delta(\alpha, \beta)) = \Xi_\alpha(\Delta(\alpha, \beta)) \neq \Xi_\beta(\Delta(\alpha, \beta)) = \Xi_{h(\gamma)}(\Delta(\alpha, \beta)).$$

We conclude using the part (a) of Lemma 3.31 that $\Delta(\gamma, h(\gamma)) = \Delta(\alpha, \beta)$. $\square$

Proof of (b) $\Rightarrow$ (a). Suppose that $\Xi_\gamma(\Delta(\alpha, \beta)) = -1$. Then the conclusion of the part (d) of Lemma 3.31 can not hold when applied to γ, $h(\gamma)$ and $\Delta(\alpha, \beta)$. In virtue of this, $\Delta(\alpha, \beta)$ must be distinct from $\Delta(\gamma, h(\gamma))$. $\square$

$\square$

Corollary 3.34. *Let $\xi < \alpha < \beta \in X$. If $\rho(\xi, \beta) < \rho(\alpha, \beta)$ then $\rho(\xi, \alpha) < \rho(\alpha, \beta)$.*

Proof. Since ρ is an ordinal metric, $\rho(\xi, \alpha) \leq \max(\rho(\xi, \beta), \rho(\alpha, \beta)) = \rho(\alpha, \beta)$. Suppose towards a contradiction that $\rho(\alpha, \beta) = \rho(\xi, \alpha)$. Then by the part (b) of Lemma 3.31 we have that

$$0 \leq \Xi_\xi(\rho(\alpha, \beta)) < \Xi_\alpha(\rho(\alpha, \beta)) < \Xi_\beta(\rho(\alpha, \beta)).$$

On the other hand, since $\rho(\alpha, \beta) > \rho(\beta, \xi)$ then $\Xi_\xi(\rho(\alpha, \beta)) = -1$ or $\Xi_\xi(\rho(\alpha, \beta)) = \Xi_\beta(\rho(\alpha, \beta))$ due to the part (c) of Lemma 3.31. Both cases contradict the previous inequality so we are done. $\square$

The definition of the Ξ function can be extended (with some restrictions) to arbitrary elements of FIN(X). More precisely, by virtue of Proposition 3.7 we know that:

Lemma 3.35. *Let $A \in FIN(X)$, $k > \rho^A$ and $F \in \mathcal{F}_k$ such that $A \subseteq F_i$. Then there is $i < n_k$ for which $A \subseteq F$. Furthermore, it is easy to see that if $A \not\subseteq R(F)$ then this i is unique and does not depend on the choice of F.*

This leads to the following definition.

Definition 3.36 (set-valued Ξ function)**.** *Let $A \in FIN(X)$ for each $k > \rho^A$ we define*

$$\Xi_A(k) = \begin{cases} -1 & \text{if } A \subseteq R(F) \\ i & \text{if } A \not\subseteq R(F) \text{ and } A \subseteq F_i \end{cases}$$

Here, $F \in \mathcal{F}_k$ is such that $A \subseteq F$.

Remark 3.37. *It is not hard to check that actually $\Xi_A(k) = \Xi_{\max(A)}(k)$ for each $k > \rho^A$.*

The following lemma is easy to prove and it is left to the reader as it is just a variation of Lemma 3.31.

Lemma 3.38. *For any distinct $A, B \in FIN(X)$ and $k \geq \rho^{A\cup B}$ the following happens:*

(a) *If $0 \leq \Xi_A(k) \neq \Xi_B(k) \geq 0$ then $k = \rho^{A\cup B}$.*
(b) *If $k > \rho^{A\cup B}$ then $\Xi_A(k) = \Xi_B(k)$ or $\min(\Xi_A(k), \Xi_B(k)) = -1$.*

4. Theorems in ZFC

In this section, we will study a variety of uncountable structures in the framework of ZFC. Although most of these objects are well studied and several ways to prove their existence are known, we aim to show the value of construction schemes by presenting new constructions of this structures, some of which are considerably shorter than their classical counterparts.

4.1. **Countryman lines.**

Countryman lines are a certain kind of linear orders whose existence was questioned by R. S. Countryman in the 1970's. In [64], S. Shelah proved their existence from ZFC for the first time. Years later, an easy construction was found by the third author presented in [82] an easy construction of such a line using walks on ordinals. For more about Countryman lines see [26], [47], or [74]. On the other hand, Aronszajn trees are a particular kind of trees constructed by N. Aronszajn in 1930's. Their existence imply that the natural generalization of König's Lemma to ω_1 is false. For further information about Aronszajn trees, we recommend to search for [25], [34], [35], [45], [49], [50], [56], [74] or [79]. Let $\mathcal{F}$ be a construction scheme (of type $\langle m_k, n_{k+1}, r_{k+1}\rangle_{k\in\omega}$).

Definition 4.1 (Countryman line)**.** *We say that a totally ordered set $(X, <)$ is a Countryman line if X is uncountable and X^2 can be covered by countably many chains*[6]*.*

The following proposition will help by giving us a better picture of the next definitions.

Proposition 4.2. *Let $\alpha, \beta \in \omega_1$ be distinct ordinals and $k < \Delta(\alpha,\beta)$. Then $\Xi_\delta(\Delta(\alpha,\beta)) = -1$ for each $\delta \in (\alpha)_k \cap (\beta)_k$. In particular $|(\alpha)_k \cap (\beta)_k| \leq r_{\Delta(\alpha,\beta)}$.*

Proof. Let $\delta \in (\alpha)_k \cap (\beta)_k$. Suppose towards a contradiction that $\Xi_\delta(\Delta(\alpha,\beta)) \neq -1$. Thus, according to the part (c) of Lemma 3.31 and the fact that $\rho(\alpha,\delta), \rho(\beta,\delta) \leq k < \Delta(\alpha,\beta)$, we have that $\Xi_\alpha(\Delta(\alpha,\beta)) = \Xi_\delta(\Delta(\alpha,\beta)) = \Xi_\beta(\Delta(\alpha,\beta))$. This is impossible since $\Xi_\alpha(\Delta(\alpha,\beta) \neq \Xi_\beta(\Delta(\alpha,\beta))$ due to the point (d) of the same lemma. $\square$

Definition 4.3. *Let $\alpha, \beta \in \omega_1$ be distinct ordinals and $k = \Delta(\alpha,\beta) - 1$. We define c^α_β as:*

$$c^\alpha_\beta = \min(\,(\alpha)_k \backslash (\beta)_k\,)$$

[6] Here, we consider the order over X^2 given by $(x,y) \leq (w,z)$ if and only $x \leq w$ and $y \leq z$.

Note that $c^\alpha_\beta \neq c^\beta_\alpha$ for any two distinct $\alpha, \beta \in \omega_1$. Furthermore, if $k = \Delta(\alpha, \beta) - 1$, then $(c^\alpha_\beta)^-_k = (\alpha)_k \cap (\beta)_k = (c^\beta_\alpha)^-_k$. In this way, $\|c^\beta_\alpha\|_k = \|c^\alpha_\beta\|_k \leq r_{k+1}$ due to the Proposition 4.2.

Definition 4.4. *Given different $\alpha, \beta \in \omega_1$ and $k = \Delta(\alpha, \beta) - 1$, we recursively decide whether $\alpha <_\mathcal{F} \beta$ when one of the following conditions holds:*

(a) *$\|c^\alpha_\beta\|_k = r_{k+1}$ and $\Xi_\alpha(k+1) < \Xi_\beta(k+1)$.*

(b) *$\|c^\alpha_\beta\|_k < r_{k+1}$ and $c^\alpha_\beta <_\mathcal{F} c^\beta_\alpha$.*

Paraphrasing, take $E, F \in \mathcal{F}_k$ with $\alpha \in E$ and $\beta \in F$. We know $E \cap F$ is an initial segment of both. This intersection may or may not contain $R(E)$ (respectively $R(F)$). In case $R(E) \subseteq E \cap F$ (which implies $R(E) = R(F)$), we make $\alpha <_\mathcal{F} \beta$ if $|E \cap \alpha| < |F \cap \beta|$. On the other side, if $R(E) \not\subseteq E \cap F$ then we relate α and β in the same way as $\min(E \backslash F)$ and $\min(F \backslash E)$.

Lemma 4.5. *Suppose $\alpha, \beta \in \omega_1$ are distinct and let $k = \Delta(\alpha, \beta) - 1$. Consider $h : (\alpha)_k \longrightarrow (\beta)_k$ the increasing bijection and take $\gamma \in (\alpha)_k$ such that $h(\gamma) \neq \gamma$. Then the following statements are equivalent:*

(1) *$\alpha <_\mathcal{F} \beta$.*

(2) *$\gamma <_\mathcal{F} h(\gamma)$.*

Proof. The proof is carried by induction over α and β. So suppose that we have proved the Lemma for each $\alpha' < \alpha$ and $\beta' < \beta$. We start with some remarks about γ. First of all, $\Delta(\gamma, h(\gamma)) \geq \Delta(\alpha, \beta)$ due to the point (a) of Lemma 3.27. Furthermore, according to Remark 3.28 it follows that $\gamma \in (\alpha)_k \backslash (\beta)_k$ and $h(\gamma) \in (\beta)_k \backslash (\alpha)_k$. This means that $c^\alpha_\beta \in (\gamma)_k$ and $c^\beta_\alpha \in (h(\gamma))_k$. Lastly, as $(\alpha)_k \cap (\beta)_k$ is an initial segment of both $(\alpha)_k$ and $(\beta)_k$, we conclude that $(\alpha)_k \cap (\beta)_k = (\delta)_k \cap (h(\delta))_k$. We divide the rest of the proof into two cases.

<u>Case 1</u>: If $\Delta(\alpha, \beta) = \Delta(\gamma, h(\gamma))$.

Proof of case. In this case we have that $c^\alpha_\beta = c^\gamma_{h(\gamma)}$ and $c^\beta_\alpha = c^{h(\gamma)}_\gamma$. Thus, if $\|c^\alpha_\beta\|_k < r_{k+1}$ then $\alpha <_\mathcal{F} \beta$ if and only if $c^\alpha_\beta <_\mathcal{F} c^\beta_\alpha$ if and only if $\gamma <_\mathcal{F} h(\gamma)$. On the other hand, if $\|c^\alpha_\beta\|_k = r_{k+1}$ then $\alpha <_\mathcal{F} \beta$ if and only if $\Xi_\alpha(k+1) < \Xi_\beta(k+1)$, and $\gamma <_\mathcal{F} h(\gamma)$ if and only if $\Xi_\gamma(k+1) < \Xi_{h(\gamma)}(k+1)$. Now, as $k+1 = \Delta(\gamma, h(\gamma))$ then both $\Xi_\gamma(k+1)$ and $\Xi_{h(\gamma)}(k+1)$ are non-negative. From this fact and the part (c) of Lemma 3.31, we conclude that $\Xi_\alpha(k+1) = \Xi_\gamma(k+1)$ and $\Xi_\beta(k+1) = \Xi_{h(\gamma)}(k+1)$. In this way, $\alpha <_\mathcal{F} \beta$ if and only if $\gamma <_\mathcal{F} h(\gamma)$. $\square$

<u>Case 2</u>: If $\Delta(\alpha, \beta) < \Delta(\gamma, h(\gamma))$.

Proof of case. First we argue that $\alpha <_\mathcal{F} \beta$ if and only if $c^\alpha_\beta <_\mathcal{F} c^\beta_\alpha$. For this purpose it is enough to show that, in this case, $\|c^\alpha_\beta\|_k < r_{k+1}$. Indeed, according to the point (a) of Lemma 3.31, we have that $\Xi_\gamma(k+1) = \Xi_{h(\gamma)}(k+1)$. On the other hand, by the part (d) of such lemma, we also know that $\Xi_\alpha(k+1) \neq \Xi_\beta(k+1)$. Thus, $\Xi_\gamma(k+1) = -1$ due to the part (c) of Lemma 3.31. Since $c^\alpha_\beta \in (\gamma)_k$, it follows that $\|c^\alpha_\beta\|_k < r_{k+1}$. In this way, $\alpha <_\mathcal{F} \beta$ if and only if $c^\alpha_\beta <_\mathcal{F} c^\beta_\alpha$ as we wanted. The proof of this case will be over once we show that $\gamma <_\mathcal{F} h(\gamma)$ if and only if $c^\alpha_\beta <_\mathcal{F} c^\beta_\alpha$. Let $l = \Delta(\gamma, h(\gamma)) - 1$ and consider $h' : (\gamma)_l \longrightarrow (h(\gamma))_l$ be the increasing bijection. As $k < l$ then $c^\alpha_\beta \in (\gamma)_k \subseteq (\gamma)_l$. Furthermore,

$$h'(c^\alpha_\beta) = h'(\,(\gamma)_k(\|c^\alpha_\beta\|_k)\,) = (h(\gamma))_k(\|c^\alpha_\beta\|_k) = (h(\gamma))_k(\|c^\beta_\alpha\|_k) = c^\beta_\alpha.$$

As $c^\alpha_\beta \neq c^\beta_\alpha$, we may use the inductive hypotheses to conclude that $\gamma <_\mathcal{F} h(\gamma)$ if and only if $c^\alpha_\beta <_\mathcal{F} c^\beta_\gamma$. This finishes the proof. □

□

The following corollaries are direct consequences of the previous lemma.

Corollary 4.6. *let $\alpha, \beta \in \omega_1$ be distinct ordinals. Then $\alpha <_\mathcal{F} \beta$ if and only if $c^\alpha_\beta <_\mathcal{F} c^\beta_\alpha$.*

Corollary 4.7. *Suppose $\alpha, \beta \in \omega_1$ are distinct and let $k < \Delta(\alpha, \beta)$. Consider $h : (\alpha)_k \longrightarrow (\beta)_k$ the increasing bijection and take $\gamma \in (\alpha)_k$ such that $h(\gamma) \neq \gamma$. Then the following statements are equivalent:*

(1) $\alpha <_\mathcal{F} \beta$.
(2) $\gamma <_\mathcal{F} h(\gamma)$.

Proof. Let $\psi : (\alpha)_{\Delta(\alpha,\beta)-1} \longrightarrow (\beta)_{\Delta(\alpha,\beta)-1}$ be the increasing bijection. To prove the corollary, just notice that $\psi|_{(\alpha)_z} = \phi$. □

By means of Lemma 4.5 and its subsequent corollaries, it follows that $(\omega_1, <_\mathcal{F})$ is a total order.

Lemma 4.8. *Let $\alpha, \beta, \delta \in \omega_1$ be distinct ordinals and $k \in \omega$. If $(\alpha)_k \cap (\beta)_k \subseteq (\alpha)_k \cap (\delta)_k$ then $(\alpha)_k \cap (\beta)_k = (\beta)_k \cap (\delta)_k$.*

Proof. Since $(\alpha)_k \cap (\beta)_k \subsetneq (\alpha)_k \cap (\delta)_k$ then $(\alpha)_k \cap (\beta)_k = (\alpha)_k \cap (\delta)_k \cap (\beta)_k \subseteq (\delta)_k \cap (\beta)_k$. In order to prove that $(\beta)_k \cap (\delta)_k \subseteq (\alpha)_k \cap (\beta)_k$, note that both $(\alpha)_k \cap (\delta)_k$ and $(\beta)_k \cap (\delta)_k$ are initial segments of $(\delta)_k$ by virtue of the point (6) in Proposition 3.6. In this way, either $(\beta)_k \cap (\delta)_k \sqsubseteq (\alpha)_k \cap (\delta)_k$ or $(\alpha)_k \cap (\delta)_k \sqsubseteq (\beta)_k \cap (\delta)_k$. The latter alternative can not happen as it would imply that $(\alpha)_k \cap (\delta)_k \subseteq (\alpha)_k \cap (\beta)_k$. Thus, the first alternative holds. In this way, $(\beta)_k \cap (\delta)_k \subseteq (\beta)_k \cap (\alpha)_k \cap (\delta)_k \subseteq (\alpha)_k \cap (\beta)_k$. This finishes the proof. □

Proposition 4.9. *$(\omega_1, <_\mathcal{F})$ is a total order.*

Proof. The only non-trivial part of this task is to prove transitivity. This proof will be performed by induction. For this purpose let $\alpha, \beta, \delta \in \omega_1$ be distinct ordinals. Suppose that we have proved that for any $\alpha' < \alpha$, $\beta' < \beta$ and $\delta' < \delta$, the triplet $\{\alpha', \beta', \delta'\}$ do not form a cycle. That is, neither $\alpha' <_\mathcal{F} \beta' <_\mathcal{F} \delta' <_\mathcal{F}< \alpha'$ nor $\alpha' >_\mathcal{F} \beta' >_\mathcal{F}> \delta' >_\mathcal{F} \alpha'$. We will show that the same holds for $\{\alpha, \beta, \delta\}$. By virtue of Lemma 3.26, we may assume without loss of generality that $\Delta(\alpha, \beta) = \Delta(\alpha, \delta) \leq \Delta(\beta, \delta)$. Let $k = \Delta(\alpha, \beta) - 1$. For each distinct $x, y \in \{\alpha, \beta, \delta\}$ consider $e^x_y = \min(\,(x)_k \backslash (y)_k\,)$ and $h^x_y : (x)_k \longrightarrow (y)_k$ the increasing bijection. Observe that $h^z_x \circ h^y_z = h^y_x$, $h^x_y(e^x_y) = e^y_x$ and $e^x_y \neq e^y_x$. We divide the rest of the proof into the following cases.

<u>Case 1</u>: There are distinct $x, y, z \in \{\alpha, \beta, \delta\}$ for which $(x)_k \cap (y)_k \subsetneq (x)_k \cap (z)_k$.

Proof of case. By means of the Lemma 4.8, $(x)_k \cap (y)_k = (z)_k \cap (y)_k$. It then follows that $e^y_z = e^y_x$. Now we will now show that $e^x_y = e^z_y$. First recall that both $(x)_k \cap (y)_k$ and $(x)_k \cap (z)_k$ are initial segments of $(x)_k$. From this fact we can deduce that $(x)_k \cap (y)_k$ is not an initial segment of $(x)_k \cap (z)_k$. By minimality, $e^x_y \in (x)_k \cap (z)_k$. As $e^x_y \notin (y)_k$, it follows that $e^x_y \geq e^z_y$. Moreover, since $(x)_k \cap (z)_k \sqsubseteq (z)_k$ then $e^z_y \in (x)_k \cap (z)_k$. Again, we have that $e^z_y \geq e^x_y$ because $e^z_y \notin (y)_k$. Therefore, $e^z_y = e^x_y$. According to the Corollary 4.7, y and x are ordered (with respect to $<_\mathcal{F}$) in the same way as e^y_x and $h^y_x(e^y_x) = e^x_y$, and y and z are ordered in the same way as $e^y_z = e^y_x$ and $h^y_z(e^y_z) = e^z_y = e^x_y$. We conclude that either $y <_\mathcal{F} x, z$ or $x, z <_\mathcal{F} y$. This finishes the proof of this case. □

Case 2: $(\alpha)_k \cap (\beta)_k = (\alpha)_k \cap (\delta)_k = (\beta)_k \cap (\delta)_k$.

Proof of case. We will divide the proof of this case into two subcases, but first, note that $e^\alpha_\beta, e^\beta_\alpha, e^\alpha_\delta$ and e^δ_α are equal to $c^\alpha_\beta, c^\beta_\alpha, c^\alpha_\delta$ and c^δ_α respectively. Moreover, $e^x_y = e^x_z$ for all distinct $x, y, z \in \{\alpha, \beta, \delta\}$.

Subcase 1: $|(\alpha)_k \cap (\beta)_k| < r_{k+1}$.

Proof of subcase. By the remarks made at the start of this case and by Corollary 4.7, we have that α, β and δ are ordered in the same way as e^α_β, e^β_α and e^δ_α respectively. Thus, by virtue of the inductive hypotheses, it suffices to show that $e^\alpha_\beta < \alpha$, $e^\beta_\alpha < \beta$ and $e^\delta_\alpha < \delta$. Indeed, according to the hypotheses of this subcase, $\|e^x_y\|_k = |(x)_k \cap (y)_k| < r_{k+1}$ for all $x, y \in \{\alpha, \beta, \delta\}$. On the other hand, since $k+1 = \Delta(\alpha, \beta) = \Delta(\alpha, \delta) \leq \Delta(\beta, \delta)$, then $\Xi_x(k+1) \geq 0$ for each $x \in \{\alpha, \beta, \delta\}$. In other words $\|x\|_k \geq r_{k+1}$. Particularly, $e^\alpha_\beta < \alpha$, $e^\beta_\alpha < \beta$ and $e^\delta_\alpha < \delta$, so we are done. □

Subcase 2: If $|(\alpha)_k \cap (\beta)_k| = r_{k+1}$.

Proof of subcase. According to the definition of the order $<_\mathcal{F}$ and since $k+1 = \Delta(\alpha, \beta) = \Delta(\alpha, \delta)$, it follows that α and β are ordered (with respect to $<_\mathcal{F}$) in the same way as $\Xi_\alpha(k+1)$ and $\Xi_\beta(k+1)$ (with respect to the usual order), and α and δ are ordered in the same way as $\Xi_\alpha(k+1)$ and $\Xi_\beta(k+1)$. If $\Delta(\beta, \delta) > k+1$ then $\Xi_\beta(k+1) = \Xi_\delta(k+1)$ by the point (a) of Lemma 3.31. Thus, either $\alpha < \beta, \delta$ or $\beta, \delta < \alpha$. On the other hand, if $\Delta(\beta, \delta) = k+1$ then $e^\beta_\delta = c^\beta_\delta$ and e^β_δ. Therefore, β and δ are ordered in the same way as $\Xi_\beta(k+1)$ and $\Xi_\delta(k+1)$. Evidently, $\Xi_\alpha(k+1)$, $\Xi_\beta(k+1)$ and $\Xi_\delta(k+1)$ do not form a cycle, so we are done. □

There are no more subcases by virtue of the Proposition 4.2. Thus, the proof is over. □

□

Now we can prove that $(\omega_1, <_\mathcal{F})$ is a Countryman line.

Theorem 4.10 (Countryman line). $(\omega_1, <_\mathcal{F})$ *is a Countryman line.*

Proof. The only thing left to show is that ω_1^2 can be partitioned into countably many chains. For this purpose, it is sufficient to prove the same holds for $D = \{(\alpha, \beta) \in \omega_1^2 \ ; \alpha < \beta\}$. Given $x, y, z \in \omega$, define

$$P(x, y, z) = \{(\alpha, \beta) \in D \ : \ \|\alpha\|_z = x, \ \|\beta\|_z = y \ \textit{and} \ \rho(\alpha, \beta) = z\}.$$

It is clear that the family of all $P(x, y, z)$ is countable and covers D. Thus, the proof will be over once we show each $P(x, y, z)$ is a chain. Indeed, take $(\alpha, \beta), (\delta, \gamma) \in P(x, y, z)$ and suppose $\alpha <_\mathcal{F} \delta$. Since $\|\beta\|_z = \|\gamma\|_z$ then $\Delta(\beta, \gamma) > z$. In this way, we can consider h the increasing bijection from $(\beta)_z$ to $(\gamma)_z$. Note that $\alpha \in (\beta)_z$ and

$$h(\alpha) = h(\,(\beta)_z(\|\alpha\|_z)\,) = h(\,(\beta)_z(x)) = (\gamma)_z(x) = (\gamma)_z(\|\delta\|_z) = \delta.$$

So by Lemma 4.5, $\beta <_\mathcal{F} \gamma$. □

Recall that a partial order $(T, <)$ is a *tree* if $t_{\downarrow_T} = \{x \in T \ : \ x < t\}$[7] is well-ordered for any $t \in T$. In particular any tree is well-founded. Thus, according to the preliminaries, T has an associated rank function $rank_T$. Due to uniqueness, it can be shown that $rank_T(t) = ot(t_{\downarrow_T})$ for all $t \in T$. In this way, for each ordinal α, the level α of T can be described as

[7] Similarly, $t_{\uparrow_T}$.

$T_\alpha = \{ t \in T \,:\, ot(t_\downarrow) = \alpha \}$. Given an ordinal β and $t \in T$, we define $T|_\beta = \bigcup\limits_{\alpha<\beta} T_\alpha$, and $T|_t = \{s \in T \,:\, s \text{ } \textit{is comparable with } t\}$. Finally, we say that $B \subseteq T$ is a *branch* if it is a maximal chain in T.

Definition 4.11 (Aronszajn tree)**.** *Let $(T,<)$ be a tree. We say that T is an ω_1-tree if $Ht(T) = \omega_1$, and T_α is countable for each $\alpha < \omega_1$. Furthermore, if T is an ω_1-tree without uncountable branches, we call it an Aronszajn tree.*

There is a natural way to define an Aronszajn tree from a Countryman line (see [79]). In [42], the reader can find a construction (using construction schemes) of a family $\langle f_\alpha \rangle_{\alpha\in\omega_1}$ of functions satisfying that:

- $f_\alpha : \alpha \longrightarrow \omega$ for each $\alpha \in \omega_1$,
- $f_\beta|_\alpha =^* f_\alpha$ for each $\alpha < \beta \in \omega_1$.
- There is no $f : \omega_1 \longrightarrow \omega$ so that $f|_\alpha =^* f_\alpha$ for all $\alpha \in \omega_1$.[8]

Such a family naturally generates an Aronszajn tree whose elements are all the functions $g : \alpha \longrightarrow \omega$ for which there is $\beta \in \omega_1$ greater or equal to α such that $g =^* f_\beta|_\alpha$. These kind of trees are called coherent Aronszajn trees. It is a theorem of the of the third author (see [76]) that the lexicographical order of a coherent Aronszajn-tree forms a Countryman line. Another important class of trees are the so called *special*. We say that an ω_1-tree, say $(T,<)$, is called special if it can be written as a countable union of antichains. Since any chain intersect each antichain in at most one point, it follows that each special tree is Aronszajn. Coherent Aronszajn trees are sometimes special, but they might not always be (see [74]) and [25]). Let $\mathcal{F}$ be a construction scheme of type $\langle m_k, 2, r_{k+1}\rangle_{k\in\omega}$. Given $\beta \in \omega_1$, consider the function $\rho_\beta : \beta + 1 \longrightarrow \omega$ defined as $\rho_\beta(\alpha) = \rho(\alpha, \beta)$. Let

$$T = \{f \in \omega^{<\omega_1} \,:\, \exists \beta \in \omega_1 \,(dom(f) = \beta + 1 \text{ } \textit{and} \text{ } f =^* \rho_\beta)\}.$$

Then $\mathcal{F}$ is an Aronszajn tree.

4.2. **Gaps and coherent families.** Let X be a countably infinite set. We say that a family $\mathcal{T} \subseteq [X]^\omega$ is a (increasing) κ-pretower (or simply, pretower) if it is well-ordered by $\subsetneq^*$ and its order type is equal to the ordinal κ. Whenever we index a pretower with an ordered set, it is understood that this is done in an order preserving way. Note that each pretower has an associated rank function defined in terms of $\subsetneq^*$.

Definition 4.12 (Gaps and Pregaps)**.** *Let X be a countable set and $\mathcal{L}, \mathcal{R} \subseteq [X]^\omega$. We say that $(\mathcal{L}, \mathcal{R})$ is pregap, and write it as $\mathcal{L} \perp \mathcal{R}$, if $L \cap R =^* \emptyset$ for all $L \in \mathcal{L}$ and $R \in \mathcal{R}$. An element $C \in [X]^\omega$ is said to separate $(\mathcal{L}, \mathcal{R})$ if $L \subseteq^* C$ and $C \cap R =^* \emptyset$ for each $L \in \mathcal{L}$ and $R \in \mathcal{R}$. Finally, we say that $(\mathcal{L}, \mathcal{R})$ is a gap if it is a pregap and there is no $C \in [X]^\omega$ separating it. In the case in which $\mathcal{L}$ is a κ-pretower and $\mathcal{R}$ is a λ-pretower for ordinals κ and λ, we will say that $(\mathcal{L}, \mathcal{R})$ is a (κ, λ)-pregap and (κ, λ)-gap respectively.*

The existence of gaps can be deduced by a simple application of Zorn's Lemma. However, by doing this, we do not know the order type of its associated pretowers. In 1909, F. Hausdorff gave a clever construction, in ZFC, of an (ω_1, ω_1)-gap. Such a gap satisfied the following property.

[8]This condition is easily achieved by asking each f_α to be injective.

Definition 4.13 (Hausdorff condition)**.** *Let $(\mathcal{L},\mathcal{R})$ be an (ω_1,ω_1)-pregap on ω. We say that $(\mathcal{L},\mathcal{R})$ is Hausdorff if $\{R\in\mathcal{R} : rank(R) < rank(L) \text{ and } L\cap R\subseteq k\}$ is finite for each $L\in\mathcal{L}$ and $k\in\omega$.*

A slight variant is the following.

Definition 4.14 (Luzin condition)**.** *Let $(\mathcal{L},\mathcal{R})$ be an (ω_1,ω_1)-pregap on an infinite set X. We say that $(\mathcal{L},\mathcal{R})$ is Luzin if $\{R\in\mathcal{R} : rank(R)<rank(L) \text{ and } |L\cap R|\leq k\}$ is finite for each $L\in\mathcal{L}$ and $k\in\omega$.*

It is not hard to see that each Hausdorff pregap is in fact a gap, and that each Luzin pregap is Hausdorff. In [42], there is a construction of a Hausdorff gap using construction schemes. For the reader interested in knowing more about towers and gaps, we recommend him/her to look at [5], [6], [19], [54] ,[62], [74], [78], [83], [84] and [90].

Each (ω_1,ω_1)-pregap over a set X, say $(L_\alpha,R_\alpha)_{\alpha\in\omega_1}$, has an associated ω_1-pretower $\langle T_\alpha\rangle_{\alpha\in\omega_1}$ defined as $T_\alpha=L_\alpha\cup R_\alpha$ for each $\alpha\in\omega_1$. If we let f_α be the characteristic function of L_α with domain T_α, then $f_\alpha =^* f_\beta|_{T_\alpha}$ for each $\alpha<\beta\in\omega_1$. In this way, we can see that the original pregap is in fact a gap if and only if there is no $f: X\longrightarrow 2$ such that $f_\alpha =^* f|_{T_\alpha}$ for all $\alpha\in\omega_1$ (see [4], pp. 96-98). Based on these observations, various generalizations of pregaps can be formulated. The following definition was formulated by I. Farah in [17]. Previous work in this subject was done by D. E. Talayco in [70] and C. Morgan in [54].

Definition 4.15. *A coherent family of functions supported by an ω_1-tower $\langle T_\alpha\rangle_{\alpha\in\omega_1\backslash\omega}$ is a family of functions $\langle f_\alpha\rangle_{\alpha\in\omega_1\backslash\omega}$ such that:*

1. $\forall\alpha\in\omega_1\backslash\omega\,(\,f_\alpha : T_\alpha\longrightarrow\alpha\,)$,
2. $\forall\alpha,\beta\in\omega_1\backslash\omega\,(\,f_\alpha|_{T_\alpha\cap T_\beta} =^* f_\beta|_{T_\alpha\cap T_\beta}\,)$.

Given such family, we define L^ξ_α as $f^{-1}_\alpha[\{\xi\}]$ for all $\xi\in\omega_1$ and $\alpha>\xi$. Additionally, we let $\mathcal{L}^\xi=\langle L^\xi_\alpha\rangle_{\alpha>\xi}$

If $\langle f_\alpha\rangle_{\omega_1\backslash\omega}$ is as in the previous definition and $\xi<\mu\in\omega_1$, the pair $(L^\xi_\alpha,L^\mu_\alpha)_{\alpha\in\omega_1\backslash\mu}$ is a pregap. We will denote it as $(\mathcal{L}^\xi,\mathcal{L}^\mu)$ although, in principle, $\mathcal{L}^\xi=\langle L^\xi_\alpha\rangle_{\alpha>\xi}$ instead of $\langle L^\xi_\alpha\rangle_{\alpha>\mu}$.

Definition 4.16. *Let $\mathfrak{F}=\langle f_\alpha\rangle_{\alpha\in\omega_1\backslash\omega}$ be a coherent family of functions supported by an ω_1-tower. We say that $\mathfrak{F}$ is Luzin (respectively Hausdorff) if $(\mathcal{L}^\xi,\mathcal{L}^\mu)$ is Luzin (respectively Hausdorff) for each $\xi<\mu\in\omega_1$.*

In [17], a Luzin coherent family of functions supported by an ω_1-pretower was constructed in ZFC. This was achieved using forcing and appealing to Keisler's Completeness Theorem for $L^\omega(Q)$ (see [33]). Now, we present a direct construction of this object. No previous direct construction was known.

Theorem 4.17. *There is a Luzin coherent family of functions supported by an ω_1-tower.*

Proof. Let $\mathcal{F}$ be a 2-construction scheme. First we define ω_1-tower over the countable set

$$N=\bigcup_{k\in\omega}N_k$$

where each N_k is equal to $\{k\}\times k\times r_k\times r_k$. Given $\alpha\in\omega_1\backslash\omega$ we define T_α as

$$\bigcup_{\Xi_\alpha(k)\geq 0}N_k=\{(k,s,i,j)\in\omega^4\,:\,\Xi_\alpha(k)\geq 0 \text{ and } (s,i,j)\in k\times r_k\times r_k\,\}.$$

Note that for each $\alpha < \beta \in \omega_1$ and every $k > \rho(\alpha, \beta)$ we have that $T_\alpha \cap N_k \subseteq T_\beta \cap N_k$ due to the part (c) of Lemma 3.31. Consequently, $T_\alpha \subseteq^* T_\beta$. In this way $\mathcal{T} = \langle T_\alpha \rangle_{\alpha \in \omega_1 \backslash \omega}$ is a tower.

Now we define a Luzin coherent family of functions supported by $\mathcal{T}$. Let $\alpha \geq \omega$. Note that if $x \in T_\alpha$ then $x = (k, s, i, j)$ where $k > 0$, $\|\alpha\|_k > r_k$ and $i, j < r_k$ (In particular, $(\alpha)_k(i)$ and $(\alpha)_k(j)$ are defined). In this way, we can define $f_\alpha : T_\alpha \longrightarrow \alpha$ as follows:

$$f_\alpha(k, s, i, j) = \begin{cases} (\alpha)_k(i) & \textit{if } \Xi_\alpha(k) = 0 \\ (\alpha)_k(j) & \textit{if } \Xi_\alpha(k) = 1 \end{cases}$$

By definition, the family $\langle f_\alpha \rangle_{\alpha \in \omega_1 \backslash \omega}$ satisfies the point (1) of Definition 4.15. The two following claims will finish the proof.

<u>Claim 1</u>: $\langle f_\alpha \rangle_{\alpha \in \omega_1 \backslash \omega}$ satisfies the point (2) of Definition 4.15.

Proof of claim. Let $\alpha < \beta \in \omega_1 \backslash \omega$ and $(k, s, i, j) \in T_\alpha \cap T_\beta$ be such that $k > \rho(\alpha, \beta)$. By definition of T_α and T_β it follows that both $\Xi_\alpha(k)$ and $\Xi_\beta(k)$ are non-negative numbers. In virtue of the part (c) of Lemma 3.31, it must happen that $\Xi_\alpha(k) = \Xi_\beta(k)$. Furthermore, $(\alpha)_k \sqsubseteq (\beta)_k$ so $(\alpha)_k(i) = (\beta)_k(i)$ and $(\alpha)_k(j) = (\beta)_k(j)$. Hence, $f_\alpha(k, s, i, j, s) = f_\beta(k, s, i, j)$. Since all but finitely many elements of $T_\alpha \cap T_\beta$ have their first coordinate bigger than $\rho(\alpha, \beta)$, we have shown that $f_\alpha|_{T_\alpha \cap T_\beta} =^* f_\beta|_{T_\alpha \cap T_\beta}$. □

<u>Claim 2:</u> $\langle f_\alpha \rangle_{\alpha \in \omega_1 \backslash \omega}$ is Luzin.

Proof of claim. Let $\xi < \mu \in \omega_1$, $\beta > \mu$ and $n \in \omega$. We need to prove that the set

$$\{\alpha \in \beta \backslash \mu \,:\, |L^\xi_\alpha \cap L^\mu_\beta| \leq n\}$$

is finite. We claim that this set is contained in $(\beta)_l$ where $l = \max(n, \rho^{\{\xi, \mu, \beta\}})$. Indeed, take an $\alpha \in \beta \backslash \mu$ such that $\alpha \notin (\beta)_l$ and let $k = \rho(\alpha, \beta)$ (so obviously $l < k$). According to the part (b) of Lemma 3.31, $\Xi_\alpha(k) = 0$ and $\Xi_\beta(k) = 1$. Furthermore, since $\alpha \notin (\beta)_l$ then $k \geq \rho^{\{\xi, \mu, \beta\}}$. This means that both ξ and μ belong to $(\alpha)_k$. Hence

$$\Xi_\xi(k) \leq \Xi_\mu(k) \leq \Xi_\alpha(k) = 0 < \Xi_\beta(k).$$

By virtue of the part (c) of Lemma 3.31, $\Xi_\mu(k) = \Xi_\xi(k) = -1$. In other words, $\|\xi\|_k$ and $\|\mu\|_k$ are numbers strictly smaller than r_k. Moreover, $(\alpha)_k(\|\xi\|_k) = (\beta)_k(\|\xi\|_k) = \xi$ and $(\alpha)_k(\|\mu\|_k) = (\beta)_k(\|\mu\|_k) = \xi$. From this it follows that

$$\{k\} \times k \times \{\|\xi\|_k\} \times \{\|\mu\|_k\} \subseteq f^{-1}_\alpha[\{\xi\}] \cap f^{-1}_\beta[\{\mu\}] = L^\xi_\alpha \cap L^\mu_\beta.$$

As this set has cardinality k (which is bigger than n) we are done. □

□

5. Beyond ZFC

In this section we will be interested in construction schemes for which many pairs of ordinals α, β satisfy equality $\Delta(\alpha, \beta) = \rho(\alpha, \beta)$. Namely, capturing construction schemes. Although these sort of schemes have already been defined in [77], the analysis we provide is explicitly written for the first time here and it will be necessary in order to prove that the existence of fully capturing construction schemes follows from $\Diamond$.

Definition 5.1. *Let $A, B \in FIN(X)$ be such that $\rho^A, \rho^B < \rho^{A \cup B}$. We say that A and B are strongly ρ-isomorphic if for $l = \rho^{A \cup B}$ it happens that:*

- $|(A)_{l-1}| = |(B)_{l-1}|$. *That is,* $(A)_{l-1}$ *is* ρ*-isomorphic to* $(B)_{l-1}$*,*
- $h[A] = B$ *where* h *is the increasing bijection from* $(A)_{l-1}$ *to* $(B)_{l-1}$*.*

The proof of the following lemma is easy.

Lemma 5.2. *If* A *and* B *are strongly* ρ*-isomorphic, then* $\rho^A = \rho^B$ *and* $(A)_k$ *is isomorphic to* $(B)_k$ *for each* $k < \rho^{A\cup B}$*.*

The following results gives us a better picture of how two strongly ρ-isomorphic sets look inside the closure of their union. Its proof is left to the reader.

Proposition 5.3. *Let* $A, B \in FIN(X)$ *be such that* $\max(\rho^A, \rho^B) < \rho^{A\cup B} = l$*. Then* A *and* B *are strongly* ρ*-isomorphic if and only if for any* $F \in \mathcal{F}_l$ *with* $A \cup B \subseteq F$ *the following happens:*

- $\Xi_A(l), \Xi_B(l) \neq -1$.
- $h[A] = B$ *where* h *is the increasing bijection from* $F_{\Xi_A(l)}$ *to* $F_{\Xi_A(l)}$*.*

Remark 5.4. *The two conditions imposed to* F *in the previous proposition are equivalent to the existence of distinct* $i, j < n_l$ *(namely,* $\Xi_A(l)$ *and* $\Xi_B l$*) and a unique* $S \subseteq m_{l-1}$ *such that* $F_i[S] = A$ *and* $F_j[S] = B$ *(such* S *being* $F^{-1}_{\Xi_A(l)}[A]$*).*

Lemma 5.5. *Let* $A, B \in FIN(X)$ *be strongly* ρ*-isomorphic and let* $l = \rho^{A\cup B}$*. If* $F \in \mathcal{F}_l$ *is such that* $A \cup B \subseteq F$*, then* $A \cap B = A \cap R(F)$*. In particular,* $A \cap B \sqsubseteq A$ *and* $\Xi_{A\cap B}(l) = -1$*.*

Proof. Let $h : F_{\Xi_A(l)} \longrightarrow F_{\Xi_B(l)}$ be the increasing bijection. The inclusion from left to right is clear since $A \cap B = A \cap (A \cap B) \subseteq A \cap (F_{\Xi_A(l)} \cap F_{\Xi_B(l)}) = A \cap R(F)$. To prove the other one just note that $h|_{R(F)}$ is the identity function. Hence, if $\alpha \in A \cap R(F)$ then $\alpha = h(\alpha) \in B$ due to the second point of Proposition 5.3. $\square$

As a corollary we have:

Corollary 5.6. *Let* $A, B \in FIN(X)$ *be strongly* ρ*-isomorphic and let* $l = \rho^{A\cup B}$*. If* $F \in \mathcal{F}_l$ *is such that* $A \cup B \subseteq F$ *and* $A \cap B = \emptyset$ *then* $A \subseteq F_{\Xi_A(l)} \backslash R(F)$ *and* $B \subseteq F_{\Xi_A(l)} \backslash R(F)$*. In particular,* $\Xi_\alpha(l) = \Xi_A(l)$ *and* $\Xi_\beta(l) = \Xi_B(l)$ *for any* $\alpha \in A$ *and* $\beta \in B$*.*

Remark 5.7. *In virtue of Proposition 5.3, if* $A = \{\alpha\}$ *and* $B = \{\beta\}$ *for* α *and* β *distinct ordinals, then* A *and* B *are strongly* ρ*-isomorphic if and only if* $\Delta(\alpha, \beta) = \rho(\alpha, \beta)$*.*

Definition 5.8 (Captured families)**.** *Let* $\mathcal{D}$ *be a finite subset of* $FIN(X)$ *and* $l \in \omega$*. We say that that* $\mathcal{D}$ *is captured at level* l *if* $2 \leq |\mathcal{D}| \leq n_l$ *and:*

1. $l = \rho^{\cup\mathcal{D}} > \rho^D$ *for each* $D \in \mathcal{D}$*,*
2. $\{\Xi_D(l) : D \in \mathcal{D}\} = \{0, \dots, |\mathcal{D}| - 1\} = |\mathcal{D}|$.[9]
3. *For any distinct* $D, E \in \mathcal{D}$*,* D *is strongly* ρ*-isomorphic to* E*.*

Equivalently, for any $F \in \mathcal{F}_l$ *with* $\bigcup \mathcal{D} \subseteq F$ *there is* $S \subseteq m_{l-1}$ *such that*

$$\mathcal{D} = \{F_i[S] : i < |\mathcal{D}|\}.$$

Additionally, if $|\mathcal{D}| = n_l$ *we say that* $\mathcal{D}$ *is fully captured at level* l*. Whenever* $D \in FIN(X)$*, we say that* D *is captured (resp. fully captured) in case* $\{\{\alpha\} : \alpha \in D\}$ *is captured (resp. fully captured). Finally, if* $\mathcal{D}$ *is captured at level* l *and we write* $\mathcal{D}$ *as a list of elements, say* $\mathcal{D} = \{D_0, D_1, \dots, D_{n-1}\}$*, we always assume that* $\Xi_{D_i}(l) = i$ *for each* $i < n$*.*

[9]by Lemma 3.38, this condition implies that $\rho^{E\cup D} = l$ for any distinct $D, E \in \mathcal{D}$.

We will frequently use the following lemma without any explicit mention to it.

Lemma 5.9. *Let $\mathcal{D}$ be a finite subset of FIN(X) which is captured at some level $l \in \omega$. If n is the common cardinality of the members of $\mathcal{D}$ and S is a non-empty finite subset of n, then*

$$\{D[S] \,:\, D \in \mathcal{D}\}$$

is also captured at level l as long as this set has at least two elements.

The following proposition is a direct consequence of the Remark 5.7.

Proposition 5.10. *Let $l \in \omega$ and $D \in$ FIN(X) be such that $|D| \geq 2$. Then D is captured at level l if and only if:*

(1) *For any distinct $\alpha, \beta \in D$, $\Delta(\alpha, \beta) = l = \rho(\alpha, \beta)$,*
(2) *For each $i < |D|$, $\Xi_{D(i)}(l) = i$. In other words, $\{\, \Xi_\alpha(l) \,:\, \alpha \in D \,\} = |D|$.*

Definition 5.11 (n-capturing schemes)**.** *Let $\mathcal{P}$ be a partition of ω and $n \in \omega$. We say that $\mathcal{F}$ is n-$\mathcal{P}$-capturing if for each uncountable $\mathcal{S} \subseteq$ FIN(X) and $P \in \mathcal{P}$ there are infinitely many $l \in P$ with some $\mathcal{D} \in [\mathcal{S}]^n$ which is captured at level l. Whenever $\mathcal{P} = \{\omega\}$, we simply say that $\mathcal{F}$ is n-capturing.*

Definition 5.12 (capturing schemes)**.** *Let $\mathcal{P}$ be a partition of ω. We say that $\mathcal{F}$ is $\mathcal{P}$-capturing (resp. capturing) if $\mathcal{F}$ is n-$\mathcal{P}$-capturing (resp. n-capturing) for each $n \in \omega$.*

Definition 5.13 (fully capturing schemes)**.** *Let $\mathcal{P}$ be a partition of ω. We say that $\mathcal{F}$ is $\mathcal{P}$-fully capturing if for each uncountable $\mathcal{S} \subseteq$ FIN(X) and $P \in \mathcal{P}$ there are infinitely many $l \in P$ with some $\mathcal{D} \in$ FIN$(\mathcal{S})$ which is fully captured at level l. Whenever $\mathcal{P} = \{\omega\}$, we simply say that $\mathcal{F}$ is fully capturing.*

The following Lemma was first proved in [77] (Lemma 7.1) and it presents useful equivalences of the previous definitions.

Lemma 5.14. *Let $\mathcal{F}$ be a construction scheme and $\mathcal{P}$ be a partition of ω compatible with τ. Then:*

- *For each $n \in \omega$, $\mathcal{F}$ is n-$\mathcal{P}$-capturing if and only if for each $S \in [X]^{\omega_1}$ and $P \in \mathcal{P}$ there is $D \in [S]^n$ which is captured at some level $l \in P$.*
- *$\mathcal{F}$ is $\mathcal{P}$-fully capturing if and only if for each $S \in [X]^{\omega_1}$ and $P \in \mathcal{P}$ there is $D \in$ FIN(S) which is fully captured at some level $l \in P$.*

We are now ready to state the Capturing axioms. The axiom $FCA(part)$ was introduced in [77]. The other axioms were later studied in [23], [31], [41] and [42].

Fully Capturing Axiom [FCA]: There is a fully capturing construction scheme over ω_1 of every possible good type.

Fully Capturing Axiom with Partitions [FCA(part)]: There is a $\mathcal{P}$-fully capturing construction scheme over ω_1 for every good type τ and each partition $\mathcal{P}$ compatible with τ.

n-Capturing Axiom [CA$_n$]: For any $n' \leq n$, there is an n'-capturing construction scheme over ω_1 of every possible good type satisfying that $n' \leq n_k$ for each $k \in \omega \backslash 1$.

n-Capturing Axiom with Partitions [CA$_n$(part)]: For any $n' \leq n$, there is a $\mathcal{P}$-n'-capturing construction scheme over ω_1 for every good type τ satisfying that $n' \leq n_k$ for each $k \in \omega \backslash 1$ and each partition $\mathcal{P}$ compatible with τ.

Capturing Axiom [CA]: CA_n holds for each $n \in \omega$ and there is a capturing construction scheme over ω_1 for every good type satisfying that the sequence $\langle n_{k+1} \rangle_{k \in \omega}$ is non-decreasing and unbounded.

Capturing Axiom with partitions [CA(part)]: $CA_n(part)$ holds for each $n \in \omega$ and there is a $\mathcal{P}$-capturing construction scheme over ω_1 for every good type τ satisfying that the sequence $\langle n_{k+1} \rangle_{k \in \omega}$ is non-decreasing and unbounded and each partition $\mathcal{P}$-compatible with τ.

5.1. **Types of Suslin trees.** In this subsection we study a particular class of Aronszajn trees called Suslin. T. Jech and S. Tennenbaum independently proved the consistency of the existence of Suslin trees in [29] and [71] respectively. Later, R. Jensen proved that the existence of Suslin trees follows from the $\diamondsuit$-principle. Finally, R. M. Solovay and S. Tennenbaum proved that ZFC is consistent with the non existence of Suslin trees (see [67]).

Definition 5.15. *Let $(T, <)$ be an ω_1-tree. we say that T is Suslin if $(T, >)$ does not contain uncountable chains or antichains.*

Definition 5.16 (Coherent tree)**.** *Let $(T, <)$ be a Suslin tree. We say that T is coherent if there is a family of functions $\langle f_\alpha \rangle_{\alpha \in \omega_1}$ so that the following properties hold for all $\alpha < \beta \in \omega_1$:*

(1) $f_\alpha : \alpha \longrightarrow \omega$,
(2) $f_\alpha =^* f_\beta|_\alpha$.

Furthermore, $T = \{ f_\alpha|_\xi \,:\, \xi \leq \alpha < \omega_1 \}$ and the order $<$ of T coincides with $\subseteq$.

Coherent trees have been extensively studied in the past, since they have many interesting forcing properties (see [7], [8], [37], [39], [40] and [45]).

Theorem 5.17 (Under FCA(part))**.** *There is a coherent Suslin tree.*

Proof. Let $\tau = \langle m_k, n_{k+1}, r_{k+1} \rangle_{k \in \omega}$ be a type so that $n_{k+1} \geq 2^{m_k - r_{k+1}} + 1$ for each $k \in \omega$. Furthermore, let $\mathcal{P} = \{P_c, P_a\}$ be a partition of ω compatible with τ. We will build a coherent Suslin tree by using a $\mathcal{P}$-fully capturing construction scheme $\mathcal{F}$ of type τ.

For each $k \in \omega$, we first enumerate (possibly with repetitions) the set of all functions from $m_k \backslash r_{k+1}$ into 2 as $\langle g_i^k \rangle_{0 < i < n_{k+1}}$. Now, given $\beta \in \omega_1$, we define $f_\beta : \beta \longrightarrow 2$ as follows:

$$f_\beta(\xi) = \begin{cases} 1 & \text{if } \Xi_\xi(l) = 0,\ \Xi_\beta(l) = 1 \text{ and } l \in P_c \\ g_i^l(\|\xi\|_l) & \text{if } \Xi_\xi(l) = 0 \text{ and } l \in P_a \\ 0 & \text{otherwise} \end{cases}$$

Here, $l = \rho(\xi, \beta)$.
Before defining the tree, we will prove that the sequence $\langle f_\alpha \rangle_{\alpha \in \omega_1}$ satisfies the condition (2) of the Definition 5.16. This will be done in the following claim.

<u>Claim 1</u>: Let $\xi < \alpha < \beta \in \omega_1$. If $\xi \notin (\alpha)_{\rho(\alpha,\beta)}$ then $f_\alpha(\xi) = f_\beta(x)$.

Proof of claim. Note that $\rho(\xi,\alpha) > \rho(\alpha,\beta)$ we the hypotheses. As ρ is an ordinal metric, we use the previous fact to conclude that $\rho(\xi,\alpha) = \rho(\xi,\beta)$. Let us call this number l. According to the part (c) of Lemma 3.31, either $\Xi_\alpha(l) = -1$ or $\Xi_\alpha(l) = \Xi_\beta(l)$. On the other hand, $0 \leq \Xi_\xi(l) < \Xi_\alpha(l)$ due to part (b) of the same lemma. In this way, $\Xi_\alpha(l) = \Xi_\beta(l)$. By definition of f_α and f_β, $f_\beta(\xi) = f_\alpha(\xi)$. □

Now we define T as expected. That is, $T = \{f_\alpha|_\xi \, : \, \xi \leq \alpha < \omega_1\}$ and consider it ordered by $\subseteq$. In the next two claims, we will prove that T is the coherent Suslin tree we are looking for.

Claim 2: T does not have uncountable chains.

Proof of claim. Let $S \in [T]^{\omega_1}$. Without loss of generality we can suppose that each element of S is of the form $f_\alpha|_\xi$ for some $\xi < \alpha \in \omega_1$. Consider

$$\mathcal{C} = \{\, C \in [\omega_1]^2 \, : \, f_{C(1)}|_{C(0)} \in S\}.$$

By refining $\mathcal{C}$, we may assume that its elements are pairwise disjoint. Since $\mathcal{F}$ is $\mathcal{P}$-3-capturing, there are distinct $C_0, C_1, C_2 \in \mathcal{C}$ so that $\{C_0, C_1, C_2\}$ is captured at some level $l \in P_c$. For convenience, let us denote $C_i(0)$ and $C_i(1)$ simply as ξ_i and α_i for each $i < 3$. In order to finish, just note that $\Xi_{\alpha_1}(l) = 1$, $\Xi_{\alpha_2}(l) = 2$ and $\rho(\xi_0,\alpha_1) = \rho(\xi_0,\alpha_2) = l$. Thus, according to the definition, we have that $f_{\alpha_1}(\xi_0) = f_{\alpha_1}|_{\xi_1}(\xi_0) = 1$ and $f_{\alpha_2}(\xi_0) = f_{\alpha_2}|_{\xi_2}(\xi_0) = 0$. Hence, S is not a chain. □

Claim 3: T has no uncountable antichains.

Proof of claim. Let $A \in [T]^{\omega_1}$. Without loss of generality we can suppose that each element of A is of the form f_α for some $\alpha \in \omega_1$. Consider

$$\mathcal{D} = \{\alpha \in \omega_1 \, : \, f_\alpha \in A\}.$$

Since $\mathcal{F}$ is $\mathcal{P}$-fully capturing, there is $D \in \mathrm{FIN}(\mathcal{D})$ which is fully-captured at some level $l \in P_a$. For convenience, let us denote $D(i)$ as α_i for each $i < l$. Now, let $g : m_{l-1} \backslash r_l \longrightarrow 2$ be given by:

$$g(j) = \begin{cases} f_{\alpha_0}(\,(\alpha_0)_l(j)\,) & \text{if } j < \|\alpha_0\|_l \\ 0 & \text{otherwise} \end{cases}$$

Take $0 < i < l$ for which $g = g^l_i$. We claim that $f|_{\alpha_0} = f_{\alpha_i}|_{\alpha_0}$. For this purpose, take an arbitrary $\xi < \alpha_0$. If $\xi \in (\alpha_0)_l$ and $r_l \leq \|\xi\|_l$ then

$$f_{\alpha_i}(\xi) = g^l_i(\|\xi\|_l) = g(\|\xi\|_l) = f_{\alpha_0}((\alpha_0)_l(\|\xi\|_l)) = f_{\alpha_0}(\xi).$$

If $\xi \in (\alpha_0)_l$ and $\|\xi\|_l < r_l$ then $\xi \in (\alpha_0)_l \cap (\alpha_i)_l$. By means of the Proposition 5.10, we know that $\rho(\alpha_0,\alpha_i) = \Delta(\alpha_0,\alpha_i) = l$. In other words, α_0 and α_i are strongly ρ-isomorphic (see Remark 5.7). From this, it follows that $\rho(\xi,\alpha_0) = \rho(\xi,\alpha_i)$. Thus, $f_{\alpha_i}(\xi) = f_{\alpha_0}(\xi)$ by virtue of the part (a) of Lemma 3.31 (since $\Delta(\alpha_0,\alpha_i) = l$). The last case happen when $\xi \notin (\alpha)_l$. Here, $f_{\alpha_0}(\xi) = f_{\alpha_i}(\xi)$ as a direct consequence of the Claim 1. □

□

Suppose T is a coherent Suslin tree and G is a generic filter over T (seen as a forcing notion). In the generic extension, T has exactly ω_1 uncountable branches. Indeed, the function $\bigcup G : \omega_1 \longrightarrow \omega$ determines an uncountable branch through T. Using that T is coherent, it is not hard to see that every finite modification of $\bigcup G$ has the same property. The underlying reason for this is that, in the ground model, T has many nontrivial automorphisms.

Our next goal is to construct Suslin trees which are substantially different from the coherent ones. These trees not only have a trivial automorphism group, but in fact, only one uncountable branch is added when we force with them. The reader interested in knowing more about the automorphisms of Suslin trees is invited to look for [20].

Definition 5.18 (Tree product). *Given $k \in \omega$ and $\{(T_i, <_i)\}_{i<k}$ a family of trees we define their tree product*

$$\bigotimes_{i<k} T_i = \{t \in \prod_{i<k} T_i \,:\, \forall i < k\,\big(rank(t(0)) = rank(t(i))\big)\}.$$

To this set, we associate a canonical order given by:

$$s < t \textit{ if and only if } s(i) <_i t(i) \textit{ for each } i < k$$

It is not hard to see that $\bigotimes\limits_{i<k} T_i$ is always a tree. Furthermore, the tree product of Aronszajn trees is always Aronszajn. Unfortunately, we can not say the same about the tree product of Suslin trees. In fact, the tree product of a Suslin tree with itself is never Suslin.

Definition 5.19 (Full Suslin tree). *Let $(T, <)$ be a tree. We say that T is full Suslin if $\bigotimes\limits_{i<k} T|_{t_i}$ is Suslin for every distinct $t_0, \dots, t_{k-1} \in T$, all of the same rank.*

Theorem 5.20 (Under FCA). *There is a full Suslin tree.*

Proof. Fix a type $\langle m_k, n_{k+1}, r_{k+1}\rangle_{k\in\omega}$ such that $n_{k+1} \geq 2^{m_k}$ for each $k \in \omega$, and let $\mathcal{F}$ be a fully capturing construction scheme of that type. The plan is define an order $<_T$ over ω_1 which turns it into a full Suslin tree. For this purpose, we will recursively define for each $F \in \mathcal{F}$, a tree ordering $\prec_F$. We will ask that the following conditions hold for any two $F, G \in \mathcal{F}$:

(1) For each $\alpha, \beta \in F$, if $\alpha \prec_F \beta$ then $\alpha < \beta$ in the usual ordering over ω_1.
(2) If $F \subseteq G$, then $\prec_F = \prec_G \cap (F \times F)$.
(3) If $\rho^F = \rho^G$ and $h : F \longrightarrow G$ is the increasing bijection, then h is an isomorphism between $(F, \prec_F)$ and $(G, \prec_G)$.

We proceed to define the orderings by recursion over $k = \rho^F$.

Base step: If $k = 0$, there is nothing to do. This is because $|F| = 1$ in this case.

Recursive step: Suppose that $0 < k \in \omega$ and we have constructed the required orderings over each element of $\mathcal{F}_{k-1}$. Let $F \in \mathcal{F}_k$. For each $i < n_k$ let $h_i : F_0 \longrightarrow F_i$ be the increasing bijection.

Given $I \subseteq F_0 \backslash R(F)$, let us say that I is *k-independent* if for any two distinct $\alpha, \beta \in I$ and each $\xi \in F_0$, if $\xi \prec_{F_0} \alpha, \beta$ then $\xi \in R(F)$. Note that both the empty set and singletons are k-independent. Now, let us enumerate (possibly with repetitions) the set of all k-independent sets of $F_0 \backslash R(F)$ as $\langle I^F_i\rangle_{0<i<n_k}$. Given $\alpha, \beta \in F$, we decide whether $\alpha \prec_F \beta$ if one of the two following cases occur:

- Both α and β belong to the same F_i and $\alpha \prec_{F_i} \beta$.
- $\alpha \in F_0 \backslash R(F)$, $\beta \in F_i \backslash R(F)$ for some $i > 0$ and there is a (necessarily unique) $\delta \in I^F_i$ so that $\alpha \preceq_{F_0} \delta$ and $h(\delta)$ is comparable with β.

It is straightforward to check that $\prec_F$ is a tree ordering which satisfies the condition (2) of the recursive hypotheses. Furthermore, in the definition we are only adding new relations between elements of $F_0 \backslash R(F)$ and $F_i \backslash R(F)$ for $i > 0$, it also follows that the condition (1) of the recursive hypotheses hold. It should be also clear that condition (3) holds if we choose the enumerations $\langle I_i^F \rangle_{0<i<n_k}$ always in "isomorphic" positions. This finishes the construction.

We now define $\prec = \bigcup_{F \in \mathcal{F}} \prec_F$. According to the points (1), (2) and (3) of the recursion hypotheses, it follows that $(\omega_1, <_T)$ is in fact a tree. Moreover, $\prec \cap (F \times F) = \prec_F$ for any $F \in \mathcal{F}$. We proceed to show that it is full Suslin.

Claim 1: $(\omega_1, <_T)$ does not have uncountable chains.

Proof of claim. Let $A \in [\omega_1]^{\omega_1}$. As $\mathcal{F}$ is full capturing there is $l \in \omega$ and $D \in [A]^l$ which is captured at level l. Consider $F \in \mathcal{F}_l$ so that $D \subseteq F$. $I = \emptyset$ is l-independent. Hence, there is $0 < i < n_l$ so that $I = I_i^F$. By definition of $\prec_F$, we have that $D(0)$ is incompatible with $D(1)$. So the claim is over. $\square$

Claim 2: Let $1 \leq k \in \omega$ and $t_0 < \cdots < t_{k-1} \in \omega_1$ be distinct elements of the same rank. Then $\bigotimes_{i<k} \omega_1|_{t_i}$ has no uncountable antichains.

Proof of claim. Let $\mathcal{A}$ be an uncountable subset of $\bigotimes_{i<k} \omega_1|_{t_i}$. We may assume without loss of generality that $t_{k-1} < s(0) < \cdots < s(k-1)$ for any $s \in \mathcal{A}$. Furthermore, we may suppose that if s and t are distinct elements of $\mathcal{A}$, either $s(k-1) < t(0)$ or $t(k-1) < s(0)$. Given $s \in \mathcal{A}$, let

$$D_s = \{t_0, \ldots, t_{k-1}, s(0), \ldots, s(k-1)\}.$$

As $\mathcal{F}$ is fully capturing, there $s_0, \ldots, s_{l-1} \in \mathcal{A}$ so that $\mathcal{D} = \{D_{s_0}, \ldots, D_{s_{n_{l-1}}}\}$ is fully captured at some level $l \in \omega$. Let $F \in \mathcal{F}_l$ for which

$$\bigcup D_{s_i} \subseteq F.$$

Then $D_{s_i} \subseteq F_i$ for any $i < n_l$. Moreover, $F_i \cap R(F) = \{t_0, \ldots, t_{k-1}\}$. We will show that there is $i < n_l$ so that $s_0 < s_i$ with respect to the tree product ordering. For this aim, we need the following subclaim:

Subclaim 1: $I = \{s_0(0), \ldots, s_1(k-1)\}$ is an l-independent subset of $F_0 \backslash R(F)$.

Proof of subclaim. Let $i < j < k$ and $\alpha \in F_0$ be such that $\alpha \prec s_0(i), s_0(j)$. We know that $t_i \prec s_0(i)$ and $t_j \prec s_0(j)$. Furthermore, t_i and t_j are incomparable because they are different elements of the same rank. Hence, it must happen that $\alpha \prec t_i, t_j$. By the point (1) of the recursive conditions, this implies that $\alpha < t_i, t_j$. As $t_i \in R(F)$, then $\alpha \in R(F)$ too. This completes the proof. $\square$

By virtue of the previous subclaim, there is $i < n_l$ for which $I_i^F = I$. Let us consider $h_i : F_0 \longrightarrow F_i$ the increasing bijection. Given $j < k$, we have that $s_0(j) \prec s_i(j)$ because there is $\delta \in I_i^F$, namely $s_0(j)$ so that $s_0(j) \prec \delta$ and $h(\delta) = s_i(j)$ is comparable with $s_i(j)$. In this way, we conclude that $s_0 < s_j$ with respect to the tree product ordering. This finishes the proof. $\square$

$\square$

Originally, it was proved in [43] that the existence of a 3-capturing construction scheme implies the existence of a Suslin tree. Unfortunately, there is a gap in the proof. This leaves us with the following question.

Problem 5.21. *Does the existence of an n-capturing construction scheme imply the existence of Suslin trees for some $n \in \omega \backslash \{0\}$?*

In general it would be interesting to know which kind of Suslin trees can be constructed with different forms of capturing.

5.2. **Suslin lower semi-lattices.** We say that $(L, <, \wedge)$ is a lower semi-lattice whenever $(L, <)$ is a partial order such that $\sup\{z \in X : z \leq x \textit{ and } z \leq y\}$ exists and it is equal to $x \wedge y$ for each $x, y \in L$.

5.3. **Suslin lattices.** Recall that a pie in a partial order X is a set of pairwise incomparable elements.

Definition 5.22. *Let $(X, <, \wedge)$ be a lower semi-lattice. We say that $(X, <, \wedge)$ is Suslin if*
- *$(X, <)$ is well founded,*
- *X is uncountable,*
- *X does not contain any uncountable chain nor an uncountable pie.*

Suslin lower semi-lattices were first studied by Stephen J. Dilworth, Edward Odell and Bünyamin Sari in [11], in the context of Banach spaces. In [58], Dilip Raghavan and Teruyuki Yorioka proved that, assuming the $\diamondsuit$-principle, there is a Suslin lower semi-lattice $\mathbb{S}$ which is a substructure of $(\mathscr{P}(\omega), \subseteq, \cap)$ and such that $\mathbb{S}^n$ does not contain any uncountable pie for each $n \in \omega$. A partial order which satisfies this last property is said to be a *powerful pie.*
In the following theorem, we show that CA_2 is all that is needed in order to construct a Suslin lower semi-lattice with the above mentioned properties.

Theorem 5.23 (Under CA_2). *There is $\mathbb{S} \subseteq \mathscr{P}(\omega)$ such that $(\mathbb{S}, \subseteq, \cap)$ is a Suslin lower semi-lattice which is powerful pie.*

Proof. Let $\mathcal{F}$ be a 2-capturing scheme. For each $k \in \omega$, let $A_k = m_k \times 2^k$ and $U_k = \{k\} \times m_k \times 2^{k-1}$. Since the last expression has no sense when $k = 0$, we let $U_0 = \{(0, 0, 0)\}$. Also, let $\phi_k : A_k \longrightarrow A_{k+1}$ be given as:

$$\phi_k(a, b) = \begin{cases} (a, b) & \textit{if } (a, b) \in r_{k+1} \times 2^k \\ (a + (m_k - r_{k+1}), b) & \textit{in other case} \end{cases}$$

Notice that $A_k \cup \phi_k[A_k] = m_{k+1} \times 2^k$. As the final part of the preparation, let

$$N_k = \bigcup_{i \leq k} U_k.$$

Our first objective is to construct, for each $k \in \omega$, a family $\langle S^k_x \rangle_{x \in A_k} \subseteq \mathscr{P}(N_k)$ in such way that the following conditions hold for $\mathbb{S}^k = \{\emptyset\} \cup \{S^k_x : x \in A_k\}$:

(a) $(\mathbb{S}^k, \subseteq, \cap)$ is a lower semi-lattice.
(b) $\mathbb{S}^k_0 = \{\emptyset\}$ and $\mathbb{S}^k_{i+1} = \langle S^k_x \rangle_{x \in \{i\} \times 2^k}$ for all $i < m_k$ (where $\mathbb{S}^k_i$ is the set of all elements of $\mathbb{S}^k$ of rank i).
(c) For all $x \in A_k$, $S^{k+1}_x = S^k_x$ and $S^{k+1}_{\phi_k(x)} \cap N_k = S^k_x$. In particular, $S^k_x \subseteq S^{k+1}_{\phi_k(x)}$.

(d) The function $\psi_k : \mathbb{S}^k \longrightarrow \mathbb{S}^{k+1}$ given as:

$$\psi_k(x) = \begin{cases} \emptyset & if\ x = \emptyset \\ S^{k+1}_{\phi_k(y)} & if\ x = S^k_y \end{cases}$$

is an lower semi-lattice embedding for each $k \in \omega$.

The construction is carried by recursion over k.

Base step: If $k = 0$, then $A_k = \{(0,0)\}$. In this case, we let $S^0_{(0,0)} = \{(0,0,0)\}$. Trivially, all the conditions are satisfied.

Recursive step: Suppose that we have defined $\mathbb{S}^k$ for some $k \in \omega$ in such way that the conditions (a), (b), (c) and (d) are satisfied. In order to define $\mathbb{S}^{k+1}$, we first divide A_{k+1} into three quadrants:

- $C_0 = A_k$,
- $C_1 = [m_k, m_{k+1}) \times 2^k$,
- $C_2 = m_{k+1} \times [2^k, 2^{k+1})$,

Now, take an arbitrary $x \in A_{k+1}$ and consider the following cases:

(i) If $x \in C_0$, let $S^{k+1}_x = S^k_x$.

(ii) If $x \in C_2$, then $x = (a, b)$ for some $a < m_{k+1}$ and $2^k \leq b < 2^{k+1}$. In this case, let $S^{k+1}_x = \{k+1\} \times (a+1) \times \{b - 2^k\}$.

(iii) If $x \in C_1$, let $z = \phi_k^{-1}(x)$ and consider $D_x = \{b < 2^k : S^k_{(r_{k+1}, b)} \subseteq S^k_z\}$. Observe that D_x codes the elements of $\mathbb{S}^k_{r_k+1}$ which are below S^k_z. In this case, let

$$S^{k+1}_x = S^k_z \cup \big(\bigcup_{b \in D_x} \{k+1\} \times m_k \times \{b\} \big).$$

It is easy to see that $\mathbb{S}^{k+1}$ all the required conditions. This finishes the recursive construction.

Given $k \in \omega$, let us define $f_k : \omega_1 \times \omega \longrightarrow \omega^2$ by the formula $f_k(\alpha, b) = (\|\alpha\|_k, b)$. For each $(\alpha, b) \in \omega_1 \times \omega$, let

$$S_{(\alpha,b)} = \bigcup_{k > b} S^k_{f_k(\alpha,b)}.$$

Finally, let $\mathbb{S} = \{\emptyset\} \cup \{S_x : x \in \omega_1 \times \omega\}$. We will prove first that $\mathbb{S}$ is a lower semi-lattice. For this, it suffices to show that it is closed under intersections. Indeed, take $x = (\beta, b), y = (\delta, d) \in \omega_1 \times \omega$. According to the conditions (c) and (d), $S_x \cap N_k = S^k_{f_k(x)}$ and $S_y \cap N_k = S^k_{f_k(y)}$ for each $k > \max(b, d)$. Observe that $f_k(x), f_k(y) \in A_k$ where $k = \max(\rho(\beta, \delta), b, d)$. By condition (a), we know that $S^k_{f_k(x)} \cap S^k_{f_k(y)} \in \mathbb{S}^k$. If this intersection is empty, we can use condition (d) to conclude that $S_x \cap S_y = \emptyset$. On the other hand, if $S^k_{f_k(x)} \cap S^k_{f_k(y)} = S^k_{(a,c)}$ for $(a, b) \in A_k$, then $a < \|\beta\|_k$. Thus, there is $\alpha \in (\beta)_k$ for which $\|\alpha\|_k = a$. In this way, $S^k_{f_k(x)} \cap S^k_{f_k(y)} = S^k_{f_k(\alpha,c)}$. From conditions (c) and (d) we conclude that $S_x \cap S_y = S_{(\alpha,c)}$. Now, the condition (b) implies that $\mathbb{S}$ is well-founded and its rank function satisfies the following properties:

- $rank(\emptyset) = 0$,
- $rank(\alpha, b) = \alpha + 1$ if $\alpha \in \omega$,
- $rank(\alpha, b) = \alpha$ if $\alpha \in \omega_1 \backslash \omega$.

We will now prove that $\mathfrak{S}$ is Suslin. Note that we do not have to prove that $\mathbb{S}$ has no uncountable chains since it is a substructure of $(\mathscr{P}(N), \subseteq, \cap)$, where $N = \bigcup\limits_{i\in\omega} N_i$. Thus, the only thing left to do is to prove $\mathbb{S}$ is powerful pie. For this, let $n \in \omega$ and $\mathcal{A}$ be an uncountable subset of $(\omega_1 \times \omega)^n$. For each $x \in \mathcal{A}$, let (α_i^x, b_i^x) be such that $x(i) = (\alpha_i^x, b_i^x)$ for all $i < n$. Without any loss of generality we may assume that the following conditions hold for any two distinct $x, y \in \mathcal{A}$:

- $\alpha_0^x < \cdots < \alpha_{n-1}^x$.
- $\{\alpha_j^x : j < n\} \cap \{\alpha_j^y : j < n\} = \emptyset$

We also can suppose there are $b_0, \ldots, b_{n-1} \in \omega$ such that $b_i^x = b_i$ for each $i < n$ and $x \in \mathcal{A}$. Since $\mathcal{F}$ is 2-capturing, there are distinct $x, y \in \mathcal{A}$ for which $\{\langle\alpha_i^x\rangle_{i<n}, \langle\alpha_i^y\rangle_{i<n}\}$ is captured at some level $l > \max(b_i : i < n) + 1$. Note that

$$f_{l-1}(x(i)), f_{l-1}(y(i)) \in A_{l-1}$$

for each $i < n$. Now, given $i < n$ we have that $\rho(\alpha_i^x, \alpha_i^y) = \Delta(\alpha_i^x, \alpha_i^y) = l$. From this, it follows that $\|\alpha_i^x\|_{l-1} = \|\alpha_i^x\|_l$ and $\|\alpha_i^y\|_l = \|\alpha_i^x\|_{l-1} + (m_{l-1} - r_l)$. Thus, $f_{l-1}(x(i)) = f_l(x(i))$ and

$$\phi_{l-1}(f_{l-1}(x(i))) = (\|\alpha_i^x\|_{l-1} + (m_{l-1} - r_l), b_i) = (\|\alpha_i^y\|_l, b_i) = f_l(y(i)).$$

By condition (c), we conclude that $S^l_{f_l(x(i))} = S^{l-1}_{f_{l-1}(x(i))} = S^l_{f_l(y(i))} \cap N_k \subseteq S^l_{f_l(y(i))}$. By a previous argument, this means $S_{x(i)} \subseteq S_{y(i)}$ for each $i < n$. Consequently, x and y testify that $\{(S_{z(0)}, \ldots, S_{z(n-1)}) : z \in \mathcal{A}\}$ is not a pie. This finishes the proof. $\square$

For the construction given above we used a particular type. This was done in order facilitate the calculations. Nevertheless, with some minor modifications one can check that the proof works for each type.

In [31], it was proved that the existence of n-capturing construction schemes does not imply the existence of $(n+1)$-capturing construction schemes. Since the $\Diamond$-principle implies the existence of fully capturing construction schemes, Theorem 5.23 seems to be a step forward to the solution of the following problem.

Problem 5.24 ([58])**.** *Does $ZFC + CH$ imply the existence of a Suslin lower semi-lattice?*

5.4. **A result in polychromatic Ramsey theory.** Ramsey theory was born in [59]. In that paper, F. P. Ramsey wrote a lemma which is now known as Ramsey Theorem. Namely, if $k, l \in \omega$ and $c : [\omega]^k \longrightarrow l$ is an arbitrary coloring, there is $A \in [\omega]^\omega$ such that $c|_{[A]^k}$ is monochromatic. Classical Ramsey theory is concerned about results of this flavour. In an informal sense, we could say that its goal is to find order in chaos. On the contrary, polychromatic Ramsey theory is about finding chaos. An example of this can be found in [82], where the third author proved that there is a coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ such that $c[[A]^2] = \omega_1$ for each $A \in [\omega_1]^{\omega_1}$. For this, he first constructed a function $\phi : [\omega_1]^2 \longrightarrow \omega_1$ with the property that $\phi[[A]^2]$ contains a closed an unbounded subset of ω_1 for each uncountable A. It is not hard to see that if ρ is an ordinal metric, the function ϕ defined as $\phi(\alpha, \beta) = \min(\beta)_{\Delta(\alpha,\beta)} \backslash \alpha$ satisfies this property (see [74]).

Definition 5.25. *Let $c : [\omega_1]^2 \longrightarrow \omega_1$ be a coloring, $A \subseteq \omega_1$ and κ be a (possibly finite) cardinal. We say that:*

- *c is κ-bounded if $|\, c^{-1}[\{\xi\}]\, | < \kappa$ for each $\xi \in \omega_1$.*

- *A is injective if $c|_{[A]^2}$ is injective.*

The problem whether every 2-bounded coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ has an uncountable injective sets was first asked by F. Galvin in the early 1980's, who proved that CH implies a negative answer to that question. In [80], the third author prove that it is consistent, and in particular that PFA implies that every ω-bounded $c : [\omega_1]^2 \longrightarrow \omega_1$ has an uncountable injective set.
In [2], U. Abraham, J. Cummings and C. Smyth proved that it is consistent that there is a 2-bounded coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ without uncountable injective sets in any *ccc* forcing extension. After hearing this theorem, S. Friedman asked for a concrete example of a 2-bounded coloring without an uncountable injective set, but which adquires one in a *ccc* forcing extension (*ccc*-destructible). Such example was produced in [2] assuming CH and the existence of a Suslin tree. Here, we construct one using 3-capturing construction schemes.

Theorem 5.26 (CA_3)**.** *There is a coloring $c : [\omega_1]^2 \longrightarrow \omega_1$ with the following properties:*

(1) *c is 2-bounded,*
(2) *c has no uncountable injective sets,*
(3) *c is ccc-destructible.*

Proof. Let $\mathcal{F}$ be a 3-capturing construction scheme of an arbitrary type. Let $\psi : \omega_1 \times \omega \times \omega \longrightarrow \omega_1$ be a bijection. We define $c : [\omega_1]^2 \longrightarrow \omega_1$ as follows:

$$c(\alpha,\beta) = \begin{cases} \psi(\beta, \rho(\alpha,\beta), \|\alpha\|_{\rho(\alpha,\beta)}) & \textit{if } \alpha < \beta \textit{ and } \Xi_\beta(\rho(\alpha,\beta)) > 2 \\ \psi(\beta, \rho(\alpha,\beta), \|\alpha\|_{\rho(\alpha,\beta)-1}) & \textit{if } \alpha < \beta \textit{ and } \Xi_\beta(\rho(\alpha,\beta)) \leq 2 \end{cases}$$

In the following three claims we prove that c satisfies the conclusions of the theorem.

<u>Claim 1</u>: c is 2-bounded.

Proof. Let $\xi \in \omega_1$ and suppose that $\{\alpha_0,\beta_0\}, \{\alpha_1,\beta_1\}$ and $\{\alpha_2,\beta_2\}$ are elements of $c^{-1}[\{\xi\}]$. We will show that two of these pairs are equal. For this purpose, take $\beta \in \omega_1$ and $k, a \in \omega$ for which $\phi^{-1}(\xi) = (\beta, k, a)$. Given $i < 3$ we have that $\psi^{-1} \circ c(\alpha_i, \beta_i) = (\beta, k, a)$. This means that $\beta_i = \beta$ and $\rho(\alpha_i, \beta_i) = k$. Since ρ is an ordinal metric,

$$\rho(\alpha_i, \alpha_j) \leq \max(\rho(\alpha_i, \beta), \rho(\alpha_j, \beta)) = k$$

for any $i, j < 3$. In order to finish the proof of the claim we have to consider two cases. The first one is when $\Xi_\beta(k) > 2$. Here, $\|\alpha_i\|_k = a$ for each i. Particularly, $\Delta(\alpha_0, \alpha_1) > k$. But $\rho(\alpha_0, \alpha_1) \leq k$. The only way in which this is possible is if $\alpha_0 = \alpha_1$. So this case is over. The remaining case is when $\Xi_\beta(k) < 2$. According to the part (b) of Lemma 3.31, $0 \leq \Xi_{\alpha_i}(k) < \Xi_\beta(k) \leq 2$ for each i. Therefore, there are $i < j < 3$ for which $\Xi_{\alpha_i}(k) = \Xi_{\alpha_j}(k)$. Hence, by the point ($d$) of Lemma 3.31, $\Delta(\alpha, \beta) \neq k$. As $\|\alpha_i\|_{k-1} = a = \|\alpha_j\|_{k-1}$, we conclude that $\Delta(\alpha_i, \alpha_j) > k \geq \rho(\alpha_i, \alpha_j)$. In this way, $\alpha_i = \alpha_j$. □

<u>Claim 2</u>: c has no uncountable injective set.

Proof. Let $S \in [\omega_1]^{\omega_1}$. Since $\mathcal{F}$ is 3-capturing, there is $\{\alpha_0, \alpha_1, \alpha_2\} \in [S]^3$ which is captured at some level $l \in \omega$. In particular, $\Xi_{\alpha_2}(l) = 2$ and for each $i < 2$ the following properties hold:

- $\rho(\alpha_i, \alpha_2) = l$,
- $\|\alpha_i\|_{l-1} = \|\alpha_2\|_{l-1}$.

In other words, $c(\alpha_0, \alpha_2) = c(\alpha_1, \alpha_2)$. $\square$

Claim 3: c is *ccc*-destructible.

Proof of claim. Let $\mathbb{P} = \{p \in [\omega_1]^{<\omega} : p \textit{ is injective}\}$ ordered by reverse inclusion. We claim that $\mathbb{P}$ is *ccc*. Since $\mathcal{F}$ is 3-capturing (thus, 2-capturing), if $\mathcal{A} \in [\mathbb{P}]^{\omega_1}$ there are $p, q \in \mathcal{A}$, $l \in \omega$ and $F \in \mathcal{F}_l$ capturing p and q. By definition of c, it is easy to see that $p \cup q$ is injective. Hence, $\mathcal{A}$ is not an antichain, and since $\mathcal{A}$ was arbitrary, $\mathbb{P}$ is *ccc*. Finally, since $\mathbb{P}$ is *ccc* and uncountable, there is $p \in \mathbb{P}$ which forces the generic filter to be uncountable. From this it follows that if G is a $\mathbb{P}$-generic filter over V containing p, then $\bigcup G$ is an uncountable injective set. Thus, the proof is over. $\square$

$\square$

5.5. **Coherent families of functions beyond ZFC.** There is a trivial way to destroy an (ω_1, ω_1)-gap, say $(\mathcal{A}, \mathcal{B})$, via forcing. Just force with a forcing notion which colapses ω_1. In the generic extension, $(\mathcal{A}, \mathcal{B})$ will be a countable pregap, which implies that it is no longer a gap. Thus, questions about destructibility of (ω_1, ω_1)-gaps are only interesting for forcing notions which do not colapse ω_1.

Definition 5.27. *Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-pregap. We say that $(\mathcal{L}, \mathcal{R})$ is destructible if there is a forcing notion $\mathbb{P}$ which preserves ω_1 in such way that $(\mathcal{L}, \mathcal{R})$ is not a gap in some generic extension through $\mathbb{P}$. If this does not happen, the gap is said to be indestructible.*

Sometimes, it is possible to destroy one gap while making another one indestructible. Consider the following notion:

Definition 5.28. *Let $(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-pregap indexed as $(L_\alpha, R_\alpha)_{\alpha \in X}$ where X is an uncountable set of ordinals. We define the following forcing notions:*

- $\chi_0(\mathcal{L}, \mathcal{R}) = \{p \in [X]^{<\omega} : \big(\bigcup\limits_{\alpha \in p} L_\alpha\big) \cap \big(\bigcup\limits_{\alpha \in p} R_\alpha\big) = \emptyset\}$.
- $\chi_1(\mathcal{L}, \mathcal{R}) = \{p \in [X]^{<\omega} : \forall \alpha \neq \beta \in p\, \big((L_\alpha \cap R_\beta) \cup (L_\beta \cap R_\alpha) \neq \emptyset\big)\}$.

both ordered by reverse inclusion.

It turns out that the forcing notions χ_0 and χ_1 characterize when a pregap is a gap and when it is destructible respectively.

Theorem 5.29 ([9], [62], [83], [91])**.** *$(\mathcal{L}, \mathcal{R})$ be an (ω_1, ω_1)-pregap:*

- *$\chi_1(\mathcal{L}, \mathcal{R})$ is ccc if and only if $(\mathcal{L}, \mathcal{R})$ is a gap. In this case, there is some condition in $\chi_1(\mathcal{L}, \mathcal{R})$ forcing $(\mathcal{L}, \mathcal{R})$ to be indestructible.*
- *$\chi_0(\mathcal{L}, \mathcal{R})$ is ccc if and only if $(\mathcal{L}, \mathcal{R})$ is destructible. In this case, there is some condition in $\chi_0(\mathcal{L}, \mathcal{R})$ forcing $(\mathcal{L}, \mathcal{R})$ to be separated.*

Definition 5.30 (Independent families of gaps)**.** *We say that a family $\langle(\mathcal{L}^c, \mathcal{R}^c)\rangle_{c \in I}$ of (ω_1, ω_1)-pregaps is independent if:*

$$\prod_{c \in I}^{FS} \chi_{\phi(c)}(\mathcal{L}^c, \mathcal{R}^c)$$

is ccc for any $\phi : I \longrightarrow 2$. Additionally, we say that a coherent family of functions $\mathfrak{F}$ supported by an ω_1-pretower is independent if the family $\langle(\mathcal{L}^{c(0)}, \mathcal{L}^{c(1)})\rangle_{c \in [\omega_1 \setminus \omega]^2}$ is independent.

The point of the definition above is that if $\langle(\mathcal{A}_i,\mathcal{B}_i)\rangle_{i\in I}$ is an independent family of (ω_1,ω_1)-gaps, then for every $F: I \longrightarrow 2$, the finite support product

$$\prod_{i\in I}^{FS} \chi_{F(i)}(\mathcal{L}_i,\mathcal{R}_i)$$

is *ccc*. This can be used for coding in a similar way that U. Abraham an S. Shelah did in [3].

In [91], T. Yorioka proved, assuming the $\diamondsuit$-principle, that there is an independent family of 2^{ω_1} gaps. In this subsection, we will construct an independent family of gaps from FCA. For this purpose, we will need the following proposition (see [78]).

Proposition 5.31. *Let $\langle(\mathcal{L}^c,\mathcal{R}^c)\rangle_{c\in I}$ be a finite family of (ω_1,ω_1)-gaps indexed as $(L^c_\alpha,R^c_\alpha)_{\alpha\in X_c}$ for each $c\in I$. Also let $\phi: I\longrightarrow 2$. Suppose that for any uncountable $\mathcal{B}\subseteq\prod\limits_{i\in I}X_i$ there are distinct $g,h\in\mathcal{B}$ so that for any $c\in I$, if $g(c)\neq h(c)$ then:*

$$(L^c_{g(c)}\cap R^c_{h(c)})\cup(L^c_{h(c)}\cap R^c_{g(c)})=\emptyset \text{ if } \phi(c)=0,$$
$$(L^c_{g(c)}\cap R^c_{h(c)})\cup(L^c_{h(c)}\cap R^c_{g(c)})\neq\emptyset \text{ if } \phi(c)=1.$$

Then $\mathbb{P}=\prod\limits_{c\in I}\chi_{\phi(c)}(\mathcal{L}^c,\mathcal{R}^c)$ is ccc.

Theorem 5.32 (FCA)**.** *There is an independent coherent family of functions supported by an ω_1-tower.*

Proof. The construction here is similar to the one in Theorem 4.17. Fix a type $\langle m_k,n_{k+1},r_{k+1}\rangle_{k\in\omega}$ such that $n_{k+1}>2^{r^2_{k+1}}$ for all $k\in\omega$, and let $\mathcal{F}$ be a fully capturing construction scheme of that type. For each $k>0$, let

$$N_k=\{k\}\times[r_k]^2,$$
$$N=\bigcup_{k>0}N_k.$$

Also enumerate $\mathscr{P}([r_k]^2)$ (possibly with repetitions) as $\langle S^k_i\rangle_{i<n_k}$ in such way that $S^k_0=S^k_1=[r_k]^2$. We start by defining an ω_1-tower over N. Given $\alpha\geq\omega$ and $k>0$ we define $T^k_\alpha\subseteq N_k$ as follows:

$$T^k_\alpha=\begin{cases}\emptyset & \textit{if } \Xi_\alpha(k)=-1\\ \{k\}\times S^k_i & \textit{if } \Xi_\alpha(k)=i\geq 0\end{cases}$$

Finally, let $T_\alpha=\bigcup_{k\in\omega\backslash 1}T^k_\alpha$. In order to prove that $\mathcal{T}=\langle T_\alpha\rangle_{\alpha\in\omega_1\backslash\omega}$ is in fact an ω_1-tower just note that if $\alpha<\beta\in\omega_1\backslash\omega$ and $k>\rho(\alpha,\beta)$ then $T^k_\alpha\subseteq T^k_\beta$ by means of the point (c) of Lemma 3.31.

Now we will construct a coherent family of functions supported by $\mathcal{T}$. For this, let $\beta\geq\omega$. Note that if $x\in T_\beta$ then $x=(k,s)$ where $k>0$, $\Xi_k(\beta)\geq 0$ and $s\in S^k_{\Xi_\beta(k)}\subseteq[r_k]^2$. In this way, we can define $f_\beta: T_\beta\longrightarrow\beta$ as:

$$f_\beta(k,s)=\begin{cases}(\beta)_k(s(0)) & \textit{if } \Xi_\beta(k)=0\\ (\beta)_k(s(1)) & \textit{if } \Xi_\beta(k)>0\end{cases}$$

It is easy to check that $\langle f_\alpha\rangle_{\alpha\in\omega_1\backslash\omega}$ is a coherent family of functions supported by $\mathcal{T}$. This is done, again, by appealing to the point (c) of Lemma 3.31.

The only thing left to show is that $\langle f_\alpha \rangle_{\alpha\in\omega_1\backslash\omega}$ is independent. Recall that a finite support product of forcings is *ccc* if and only if each finite subproduct is *ccc*. Let I be a non-empty finite subset of $[\omega_1\backslash\omega]^2$ and let $\phi : I \longrightarrow 2$. We will finish by proving the following claim.

Claim: $\mathbb{P} = \prod\limits_{c\in I} \chi_{\phi(c)}(\mathcal{L}^{c(0)}, \mathcal{L}^{c(1)})$ is *ccc*.

Proof of claim. The proof of this claim will be performed by appealing to the equivalence provided by Proposition 5.31. First note that for any $c \in I$, $(\mathcal{L}^{c(0)}, \mathcal{L}^{c(1)}) = (L^{c(0)}_\alpha, L^{c(1)}_\alpha)_{\alpha\in X_c}$ where $X_c = \omega_1\backslash(c(1)+1)$. Now, let $\mathcal{B}$ be an uncountable subset of $\prod\limits_{c\in I} X_c$. For any $g \in \mathcal{B}$, define $D_g = \bigcup I \cup im(g)$. We need to prove the following claim.

(1) If $g, h \in B$ then $|im(g)| = |im(h)|$. Furthermore, if $\psi : im(g) \longrightarrow im(h)$ is the increasing bijection then $\psi(g(c)) = h(c)$ for any $c \in I$.
(2) $\{D_g : g \in \mathcal{B}\}$ is a root-tail-tail Δ-system with root $\bigcup I$.

Fix $k > \rho^{\cup I}$. Since $\mathcal{F}$ is a fully capturing construction scheme, there is $l > k$ and there are distinct $g_0, \dots, g_{n_l-1} \in \mathcal{B}$ so that the family $\{D_{g_0}, \dots, D_{g_{n_l-1}}\}$ is a captured at level l. According to (2) and by Lemma 5.5, $\Xi_{\cup I}(l) = -1$. In this way, the set $S = \{\, \{\|c(0)\|_l, \|c(1)\|_l\} \ : \ c \in I \text{ and } \phi(c) = 1\}$ is a subset of $[r_l]^2$. Therefore we can find $i < n_l$ so that $S^l_i = g_i$. The proof finishes with due to the subclaim. Its proof is left to the reader.

Subclaim: g_0 and g_i satisfy the hypotheses of Proposition 5.31. That is, for any $c \in I$, if $g_0(c) \neq g_i(c)$ then:[10]

$$(L^{c(0)}_{g_0(c)} \cap L^{c(1)}_{g_i(c)}) \cup (L^{c(0)}_{g_i(c)} \cap L^{c(1)}_{g_0(c)}) = \emptyset \text{ if } \phi(c) = 0,$$

$$(L^{c(0)}_{g_0(c)} \cap L^{c(1)}_{g_i(c)}) \cup (L^{c(0)}_{g_i(c)} \cap L^{c(1)}_{g_0(c)}) \neq \emptyset \text{ if } \phi(c) = 1.$$

□

□

6. Oscillation theory on construction schemes

In [83], the third author developed a very powerful "oscillation theory" and deduced very interesting theorems from it. Usually, the oscillation theory is based on an unbounded family of functions (although there are other variations, see for example [72]). Here we will develop an oscillation theory base on a bounded family of functions defined from a 2-capturing construction scheme.
Let us start by recalling the definition of oscillation.

Definition 6.1. *Let $f, g \in \omega^\omega$ and $k \in \omega$. We define the k-oscillation set of f to g as*

$$\overline{osc}_k(f,g) = \{s \in \omega\backslash k \ : \ f(s) \leq g(s) \text{ and } f(s+1) > g(s+1)\,\},$$

Additionally, we define the oscillation number of f to g as $osc_k(f,g) = |\overline{osc}_k(f,g)|$.

[10] In this case $(\mathcal{L}^c, \mathcal{R}^c) = (L^{c(0)}_\alpha, L^{c(1)}_\alpha)_{\alpha\in X_c}$. Therefore $L^c_{g_0(c)} = L^{c(0)}_{g_0(c)}$, $R^c_{g_i(c)} = L^{c(1)}_{g_i(c)}$, $L^c_{g_i(c)} = L^{c(0)}_{g_i(c)}$ and $R^c_{g_0(c)} = L^{c(1)}_{g_0(c)}$.

Remark 6.2. *Suppose that $f, g \in \omega^\omega$ are so that $f \neq^* g$. If f and g are not comparable with respect to $<^*$ then $osc_k(f,g) = \omega = osc_k(g,f)$. However, if f and g are comparable, it is not necessarily true in general that $osc(f,g) = osc(g,f)$.*

In this section we will prove that 2-capturing construction schemes can be used to define bounded families of functions whose properties reassemble the ones from unbounded families. In particular, we will be able to show that CA_2 implies a lot of things that are also implied by the hypothesis $\mathfrak{b} = \omega_1$. This is an interesting phenomenom as CA_2 is independent from the previous assumption. The reader interested in knowing about other oscillation theories may look for [72].

For the rest of this section we fix $\mathcal{F}$ a 2-capturing construction scheme of some type $\langle m_k, n_{k+1}, r_{k+1}\rangle_{k\in\omega}$. We will analyse the behaviour of the oscillation number associated to the k-cardinality functions associated to $\mathcal{F}$. Given $\alpha \in \omega_1$ let $f_\alpha : \omega \longrightarrow \omega$ be given as:

$$f_\alpha(l) = \|\alpha\|_l.$$

We define $\mathcal{B}_\mathcal{F}$ as $\langle f_\alpha\rangle_{\alpha\in\omega_1}$. The following lemma is a direct consequence of the definitions of ρ, Δ and Lemma 3.31.

Lemma 6.3. *Let $\alpha < \beta \in \omega_1$. Then:*

1. $f_\alpha(i) = f_\beta(i)$ *if* $i < \Delta(\alpha,\beta)$,
2. $f_\alpha(j) < f_\beta(j)$ *whenever* $j \geq \rho(\alpha,\beta)$,
3. $f_\alpha < f_\beta$ *provided that* $\Delta(\alpha,\beta) = \rho(\alpha,\beta)$
4. *In particular,* $f_\alpha <^* f_\beta$.

Furthermore, $\mathcal{B}_\mathcal{F}$ is bounded by the function in ω^ω which sends each i to m_i.

It is interesting that even though $\mathcal{B}_\mathcal{F}$ is bounded, its oscillation theory mirrors the oscillation theory of [83] for unbounded families. Since $\mathcal{F}$ is 2-capturing, given any $\mathcal{A} \in [\mathcal{B}_\mathcal{F}]^{<\omega}$, there are $\alpha < \beta \in \mathcal{A}$ with $\Delta(\alpha,\beta) = \rho(\alpha,\beta)$. Thus, we have the following corollary.

Corollary 6.4. *$(\mathcal{B}_\mathcal{F}, \leq)$ has no uncountable pies*[11].

Given $\alpha, \beta \in \omega_1$ and $k \in \omega$ we will write $osc_k(\alpha,\beta)$ and $\overline{osc}_k(\alpha,\beta)$ instead of $osc_k(f_\alpha, f_\beta)$ and $\overline{osc}_k(f_\alpha, f_\beta)$ respectively. These two objects will be written as $osc(\alpha,\beta)$ and $\overline{osc}(\alpha,\beta)$ whenever $k = 0$. Since osc_k is a function, $osc_\kappa[a \times b]$ stands for the set $\{osc_\kappa(a,b) : \alpha \in a \text{ and } \beta \in b\}$ whenever $a, b \subseteq \omega_1$.

Proposition 6.5. *Let $n, k \in \omega$ and $\mathcal{A} \in [\omega_1]^n$ be an uncountable family of pairwise disjoint sets such that $\rho^a = k$ for each $a \in \mathcal{A}$. Given $l \in \omega$, there are $a < b \in \mathcal{A}$ such that $osc_k[a \times b] \subseteq [l, 2l]$.*

Proof. The proof is by induction over l.

<u>Base step</u>: Suppose that $l = 0$. Since $\mathcal{F}$ is 2-capturing, there are $a < b \in \mathcal{A}$ so that the pair $\{a, b\}$ is captured at some level $s > k$. In particular, for each $\rho(a(i), b(i)) = s = \Delta(a(i), b(i))$ and $\rho(a(i), b(j)) = s$ for all distinct $i, j < k$. According to by Lemma 6.3 and the previous observation, the following properties hold for all $i, j < k$:

1. $f_{a(i)}|_{[k,s)} = f_{b(i)}|_{[k,s)}$,
2. $f_{a(i)}|_{\omega\setminus s} < f_{b(j)}|_{\omega\setminus s}$,

[11] Recall that by pie, we mean a set of pairwise incomparable elements

(3) $f_{a(i)}|_{[k,\rho^F)} < f_{a(j)}|_{[k,\rho^F)}$ provided that $i<j$.

From the previous facts, we conclude that $osc[a,b] = \{0\} = [l, 2l]$.

Inductive step: Suppose that we have proved the proposition for some $l \in \omega$ and let $\mathcal{A}$ be as in the hypotheses. Using the inductive hypotheses, we can recursively construct an uncountable $\mathcal{C} \subseteq [\mathcal{A}]^2$ of pairwise disjoint sets so that for each $\{a,b\} \in \mathcal{C}$ the following conditition hold:

- Either $a<b$ or $b<a$.[12]
- if $a<b$ then $osc_k[a \times b] \subseteq [l, 2l]$.

From this point on, whenever say that $\{a,b\} \in \mathcal{C}$ we will assume that $a<b$. Since $\mathcal{F}$ is 2-capturing, we can find an uncountable family $\mathcal{D} \subseteq [\mathcal{C}]^2$ and $r > k+1$ with the following properties:

- Whenever $\{\{a,b\},\{c,d\}\} \in \mathcal{D}$, the pair $\{a \cup b, c \cup d\}$ is captured at level r. In particular, this implies that $r > \rho^{a\cup b}$ and $a \cup b < c \cup d$.
- For each $x, y \in D$, $\bigcup x \cap \bigcup y = \emptyset$.

Using once again that $\mathcal{F}$ is 2-capturing, we can get $s > r$ and two elements of $\mathcal{D}$, say $\{\{a_0,b_0\},\{c_0,d_0\}\}, \{\{a_1,b_1\},\{c_1,d_1\}\}$, for which the pair

$$\{(a_0 \cup b_0) \cup (c_0 \cup d_0),\ (a_1 \cup b_1) \cup (c_1 \cup d_1)\}$$

is captured at level s. We will finish by proving the following claim.

Claim: $osc_k[c_0, b_1] \subseteq [l+1, 2(l+1)]$

Proof of claim. For this, take $i, j < n$. The following properties follow from Lemma 6.3:

(1) $f_{b_1(j)}|_{[k,s)} = f_{b_0(j)}|_{[k,s)}$. This is because, in particular, $\{b_0, b_1\}$ is captured at level s.
(2) $f_{c_0(i)}|_{[k,r)} = f_{a_0(i)}|_{[k,r)}$. This is due to the fact that $\{\{a,b\},\{c,d\}\} \in \mathcal{D}$. That is, the pair $\{a_0 \cup b_0, c_0 \cup d_0\}$ is captured at level r.
(3) $f_{b_1(j)}|_{\omega\backslash s} > f_{c_0(i)}|_{\omega \backslash s}$. This is true since $\rho(b_1(j), c_0(i)) = s$ and $c_0 < b_1$.
(4) $f_{b_0(j)}|_{[r,\rho^F)} < f_{c_0(i)}|_{[r,\rho^F)}$. Similarly to the previous point. This inequality holds because $\rho(b_0(j), c_0(i)) = r$ and and $b_0 < c_0$.

We will use these properties to calculate the oscillation. First observe that we can use the part (2) of Lemma 6.3 to conclude $\overline{osc}_k(a_0(i), b_0(j)) \subseteq [k, \rho^{a_0 \cup b_0})$ and

$$\overline{osc}_k(c_0(i), b_1(j)) \subseteq [k, \rho^F).$$

According the properties (1) and (2) written above and since $r > \rho^{a_0\cup b_0}$,

$$\overline{osc}_k(c_0(i), b_1(j)) \cap [k, r-1) = \overline{osc}_k(a_0(i), b_0(j)).$$

Due to properties (1), (3) and (4) we also have that $\rho^F - 1 \in \overline{osc}_k(c_0(i), b_1(j))$. In fact, properties (1) and (4) also imply that $\rho^F - 1$ is the only element in the interval $[r, \rho^F)$ which belong to $\overline{osc}_k(c_0(i), b_1(j))$. By joining all the previous observations, we get that:

$$\overline{osc}_k(a_0(i), b_0(j)) \cup \{\rho^F - 1\} \subseteq \overline{osc}_k(c_0(i), b_1(j)) \subseteq \overline{osc}_k(a_0(i), b_0(j)) \cup \{\rho^F - 1\} \cup \{r-1\}.$$

This means that $l+1 \leq osc_k(c_0(i), b_1(j)) \leq 2l+2$. Thus, the proof is complete. □

□

[12]Recall that $a<b$ means that $\max(a) < \min(b)$.

By a careful analysis of the argument of the preceding theorem, one can show that whenever $\mathcal{A} \in [\omega_1]^{\omega_1}$ and $l \in \omega$, then there are $\alpha < \beta \in \mathcal{A}$ for which $osc(\alpha, \beta) = l$. Unfortunately, this property does not hold for arbitrary uncountable families of finite sets. Nevertheless, the previous result is enough to redefine a "corrected" oscillation.
The following lemma is easy.

Lemma 6.6. *There is a partition $\langle P_n \rangle_{n \in \omega}$ of ω such that for every $k, n \in \omega$ there is $l \in \omega$ such that $[l, 2l + k] \subseteq P_n$.*

Theorem 6.7 (CA_2)**.** *There is a coloring $o : [\omega_1]^2 \longrightarrow \omega$ such that for every uncountable family $\mathcal{A} \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets and each $n \in \omega$, there are $a < b \in \mathcal{A}$ for which $\{o(\alpha, \beta) : \alpha \in a \text{ and } \beta \in b\} = \{n\}$.*

Proof. Let $\langle P_n \rangle_{n \in \omega}$ be a partition of ω as in Lemma 6.6. Let $o : [\omega_1]^2 \longrightarrow \omega$ be defined as:

$$o(\alpha, \beta) = n \textit{ if and only if } osc(\alpha, \beta) \in P_n.$$

We claim that o satisfies the conclusion of the theorem. Indeed, let $\mathcal{A}$ be an uncountable family of pairwise disjoint finite subsets of ω_1 and $n \in \omega$. By refining $\mathcal{A}$ we may suppose that there is $k \in \omega$ such that $\rho^a = k$ for every $a \in \mathcal{A}$. Let $l \in \omega$ be such that $[l, 2l + k] \subseteq P_n$. Due to Proposition 6.5, there are $a < b \in \mathcal{A}$ such that $osc_k(a, b) \subseteq [l, 2l]$. Given $\alpha \in a$ and $\beta \in b$, it is easy to see that $\overline{osc}(\alpha, \beta)$ has at most k more elements than $\overline{osc}_k(\alpha, \beta)$. In this way, $osc(\alpha, \beta) \in [l, 2l + k] \subseteq P_n$. In other words, $o(\alpha, \beta) = n$. This finishes the proof. $\square$

The existence of a coloring with the properties stated above, already implies the existence of a much more powerful coloring. As we shall mention later, such a coloring can be used to build topological spaces with important properties.

Corollary 6.8 (CA_2)**.** *There is a coloring $o^* : [\omega_1]^2 \longrightarrow \omega$ such that for all $n \in \omega$, $h : n \times n \longrightarrow \omega$ and any uncountable family $\mathcal{A} \subseteq [\omega_1]^n$ of pairwise disjoint sets, there are $a < b \in \mathcal{A}$ for which*

$$o^*(a(i), b(j)) = h(i, j)$$

for all $i, j < n$.

Proof. Let $\langle h_n \rangle_{n \in \omega}$ be an enumeration of all $h : X \longrightarrow \omega$ for which $X \subseteq \omega^{<\omega}$ is finite and its elements are pairwise incomparable. Let us call X_n the domain of h_n. Note that for each $f \in \omega^\omega$ and every $n \in \omega$ there is at most one $\sigma \in X_n$ which is extended by f. Take a coloring o satisfying the conclusion of Theorem 6.7. We define $o^* : [\omega_1]^2 \longrightarrow \omega$ as follows: Given distinct $\alpha, \beta \in \omega_1$, if there are $\sigma_\alpha, \sigma_\beta \in X_{o(\alpha,\beta)}$ for which $\sigma_\alpha \subseteq f_\alpha$ and $\sigma_\beta \subseteq f_\beta$, put

$$o^*(\alpha, \beta) = h_{o(\alpha,\beta)}(\sigma_\alpha, \sigma_\beta).$$

In any other case, let $o^*(\alpha, \beta) = 17$. In order to prove that o^* satisfies the conclusion of the corollary, let $n \in \omega$, $h : n \times n \longrightarrow \omega$ and $\mathcal{A} \subseteq [\omega_1]^n$ be an uncountable family of pairwise disjoint sets. By refining $\mathcal{A}$ we may suppose there is $k \in \omega$ with the following properties:

(1) $\forall a \in \mathcal{A} \forall i \neq j < n \, (f_{a(i)}|_k \neq f_{a(j)}|_k)$,
(2) $\forall a, b \in \mathcal{A} \forall i < n \, (f_{a(i)}|_k = f_{b(i)}|_k)$.

Fix $a_0 \in \mathcal{A}$. Let $X = \{f_{a_0(i)}|_k : i < n\}$ and define $h : X \times X \longrightarrow \omega$ as:

$$h(f_{a_0(i)}|_k, f_{a_0(j)}|_k) = h(i, j).$$

We know that there is $m \in \omega$ for which $X = X_m$ and $h = h_m$. For such m, there are $a < b \in \mathcal{A}$ for which $o(a(i), b(j)) = m$ for all $i, j < n$. For all such i and j, $f_{a_0(i)}|_k \subseteq f_{a(i)}$ and $f_{a_0(j)}|_k \subseteq f_{a(j)}$. In this way, $o^*(a(i), b(j)) = h(f_{a_0(i)}|_k, f_{a_0(j)}|_k) = h(i, j)$. So we are done. □

As an application, we get the following:

Corollary 6.9 (CA_2)**.** *ccc is not productive.*

Proof. Let o be a coloring of Theorem 6.7. For each $n \in \omega$, let

$$\mathbb{P}_n = \{p \in [\omega_1]^{<\omega_1} \,:\, \forall \alpha, \beta \in p\,(\textit{if } \alpha \neq \beta \textit{ then } o(\alpha, \beta) = n\,)\}.$$

In particular, $\mathbb{P}_0$ and $\mathbb{P}_1$ are *ccc* but $\mathbb{P}_0 \times \mathbb{P}_1$ is not. The set $\{(\alpha, \alpha) \,:\, \alpha \in \omega_1\}$ testifies this last fact. □

6.1. **Tukey order.** We say that a partial order $(D, \leq)$ is (upwards) directed if for every $x, y \in X$ there is $z \in D$ bigger than x and y.

Proposition 6.10. *$(\mathcal{B}_{\mathcal{F}}, \leq)$ is directed.*

Proof. Let $\alpha < \beta \in \omega_1$ and let $F \in \mathcal{F}_{\rho(\alpha,\beta)}$ be such that $\{\alpha, \beta\} \subseteq F$. To finish, just observe that if $\delta = \max F$ and $s \leq \rho(\alpha, \beta)$, then $F = \bigcup\{G \in \mathcal{F}_s \,|\, G \subseteq F\}$, so there is $G \in \mathcal{F}_s$ such that $G \subseteq F$ and $\delta \in G$, and it must happen that $\delta = \max G$. This implies that $f_\delta(s) = |(\{\delta\})^-_s| = |(\delta + 1) \cap G| = m_s$. Hence, $f_\delta \geq f_\alpha, f_\beta$. □

Recall the following notion for comparing directed partial orders.

Definition 6.11 ([85])**.** *Let $(D, \leq_D)$ and $(E, \leq_E)$ be directed partial orders. We say that E is Tukey below D, and write it as $E \leq_T D$ if there is $\phi : D \longrightarrow E$ such that $\phi[X]$ is cofinal in E for each cofinal $X \subseteq D$. Furthermore, we say that E is Tukey equivalent to D, and write it as $E \equiv_T D$, if $E \leq_T D$ and $D \leq_T E$.*

The study of Tukey order was initiated by J. Tukey in [85]. Among other things, he proved that the sets 1, ω, ω_1, $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$ are non Tukey equivalent when equipped with their natural orderings. In [21], J. R. Isbell showed that under CH, there is at least one directed partial order of cardinality ω_1 which is non Tukey equivalent to any of the previous mentioned. He later improved his result in [27]. In [83], the third author proved the existence of such a directed partial order under $\mathfrak{b} = \omega_1$. In [81], he proved that consistently every directed partial order of cardinality ω_1 is Tukey equivalent to one of the first five we mentioned. From now on, we will call such an order, a sixth Tukey type. The reader interested in learning more about the Tukey ordering and related topics is invited to search for [12], [13], [22], [24], [44], [57], [53], [65] and[66].

Proposition 6.12 ([85])**.** *Let $(D, \leq_D)$ and $(E, \leq_E)$ be directed partial orders.*

- *$E \leq_T D$ if and only if there is $\phi : E \longrightarrow D$ such that $\phi[X]$ is unbounded in D for each unbounded $X \subseteq E$.*
- *$E \equiv_T D$ if and only if there is a partially ordered set C in which both D and E can be embedded as cofinal subsets.*

Let $(D, \leq)$ be a directed partial order. We say $S \subseteq D$ is ω-bounded if every countable subset of S is bounded in D. The following proposition appear in [81].

Proposition 6.13. *Let $(D, \leq)$ be a directed set with $|D| = \omega_1$. Then:*

(1) *$D \leq_T 1$ if and only if D has a greatest element.*
(2) *$D \leq_T \omega$ if and only if $cof(D) \leq \omega$.*
(3) *$D \leq \omega_1$ if and only if D is ω-bounded.*
(4) *$D \leq_T \omega \times \omega_1$ if and only if D can be covered by countably many ω-bounded sets.*
(5) *$[\omega_1]^{<\omega} \leq_T D$ if and only if there is $A \in [D]^{\omega_1}$ for which every $X \in [A]^{\omega}$ is unbounded in D.*

As a consequence we get:

Corollary 6.14. *Let $(D, \leq)$ be a directed set with $|D| = \omega_1$. Then either $D \equiv_T 1$, $D \equiv_T \omega$, $D \equiv_T \omega_1$ or $\omega \times \omega_1 \leq_T D \leq_T [\omega_1]^{<\omega}$.*

The partition relation $\gamma \to (\alpha, \beta)^2_2$ stands for the following statement; For all $c : [\gamma]^2 \longrightarrow 2$, there is a 0-monochromatic[13] subset of γ of order type α or there is a 1-monochromatic subset of γ of order type β. Its negation is written as $\gamma \not\to (\alpha, \beta)^2_2$. The following theorem is due to P. Erdös and R. Rado (see [16]). It is a generalization of a well known theorem of B. Dushnik, E. Miller and P. Erdös (see [14]).

Theorem 6.15 ([16])**.** $\omega_1 \to (\omega_1, \omega + 1)^2_2$.

In [80], the third author proved that it is consistent that it is consistent to have $\omega_1 \to (\omega_1, \alpha)^2_2$ for each $\alpha < \omega_1$. On the other side, he prove in [83] that $\mathfrak{b} = \omega_1$ implies that $\omega_1 \not\to (\omega_1, \omega + 2)^2_2$. In the following theorem, we will show that the same is true under CA_2. Although a coloring testifying the negation of the previous partition relation can be extracted from some of our other constructions, we decided to include a direct proof due to its simplicity.

Theorem 6.16 (CA_2)**.** $\omega_1 \not\to (\omega_1, \omega + 2)^2_2$.

Proof. Let $\mathcal{F}$ be a 2-capturing construction scheme of an arbitrary type. Let $c : [\omega_1]^2 \longrightarrow 2$ defined as:

$$c(\alpha, \beta) = \begin{cases} 1 & \text{if } \Delta(\alpha, \beta) = \rho(\alpha, \beta) \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathcal{F}$ is 2-capturing, it is easy to see that there are no uncountable 0-monochromatic subsets of ω_1. Suppose towards a contradiction that there is a 1-monochromatic set, say X, of order type $\omega+2$. Let β and γ be the last two elements of X. and consider $\alpha \in X \backslash (\gamma)_{\rho(\beta,\gamma)}$. Since ρ is an ordinal metric, $\rho(\alpha, \beta) = \rho(\alpha, \gamma)$. In this way, $\Delta(\beta, \delta) = \rho(\beta, \delta) < \rho(\alpha, \beta) = \Delta(\alpha, \beta)$. By Lemma 3.26, we conclude that $\Delta(\alpha, \delta) = \Delta(\beta, \delta)$. This is a contradiction to X being 1-monochromatic, so we are done. $\square$

Now, we show the existence of sixth Tukey type.

Theorem 6.17. *$(\mathcal{B}_{\mathcal{F}}, \leq)$ is a sixth Tukey type.*

Proof. By Corollary 6.14, it is enough to show $\mathcal{B}_{\mathcal{F}} \not\leq_T \omega \times \omega_1$ and $[\omega_1]^{<\omega} \not\leq_T \mathcal{B}_{\mathcal{F}}$. This will be a consequence of the next two claims, and due to the points (4) and (5) of Proposition 6.13.

<u>Claim</u>: $\mathcal{B}_{\mathcal{F}}$ does not contain any uncountable ω-bounded set.

[13]Here, X is c-monochromatic if $c|_{[X]^2}$ is constant with value c.

Proof of claim. For this, we argue by contradiction. Assume there is $A \in [\omega_1]^{\omega_1}$ for which $\langle f_\beta \rangle_{\beta \in A}$ is ω-bounded. Recursively, we can build a sequence $\langle \alpha_\xi, \beta_\xi \rangle_{\xi < \omega_1}$ of pairs of countable ordinals satisfying the following properties:

(1) $\langle \alpha_\xi \rangle_{\xi \in \omega_1}$ and $\langle \beta_\xi \rangle_{\xi \in \omega_1}$ are increasing,
(2) $\beta_\xi < \alpha_\xi$ for any $\xi \in \omega_1$,
(3) $\langle \alpha_\xi \rangle_{\xi \in \omega_1} \subseteq A$,
(4) for each $\xi \in \omega_1$, f_{β_ξ} is an upper bound of the set $\{f_{\alpha_\nu} : \nu < \xi\}$.

Since $\mathcal{F}$ is 2-capturing, $\delta < \gamma$ so that $\{\{\beta_\delta, \alpha_\delta\}, \{\beta_\gamma, \alpha_\gamma\}\}$ is captured at some level $l \in \omega$. It follows that $f_{\beta_\gamma}(l-1) = f_{\beta_\delta}(l-1) < f_{\alpha_\delta}(l-1)$. But this is a contradiction since f_{β_γ} was supposed to bound f_{α_δ}. □

Claim: Let $A \in [\omega_1]^{\omega_1}$. Then $\langle f_\alpha \rangle_{\alpha \in A}$ contains an infinite bounded set.

Proof of claim. Let us define the coloring $d : [A]^2 \longrightarrow 2$ as:

$$d(\alpha, \beta) = \begin{cases} 0 & \textit{if } f_\alpha \not\leq f_\beta \textit{ and } f_\beta \not\leq f_\alpha \\ 1 & \textit{otherwise} \end{cases}$$

By Theorem 6.15, there are two possibilities: A contains a 0-monochromatic uncountable set, or A contains a 1-monochromatic set of order type $\omega + 1$. Every 0-monochromatic set is an antichain in $\mathcal{B}_{\mathcal{F}}$, so by Corollary 6.4 there can not be uncountable 0-monochromatic sets. Hence, there is 1-monochromatic subset of A, say X, of order type $\omega + 1$. Observe that f_β bounds $\langle f_\alpha \rangle_{\alpha \in X}$ where $\beta = \max(X)$. □

□

Corollary 6.18 (CA_2)**.** *There is a sixth Tukey type.*

Problem 6.19. *Does CA_2 imply the existence of 2^{ω_1} non equivalent Tukey types of size ω_1?*

6.2. **Suslin Towers.** Let $\mathcal{T}$ be an κ-pretower. We say $\mathcal{T}$ is Suslin if for every uncountable $\mathcal{A} \subseteq \mathcal{T}$ there are distinct $A, B \in \mathcal{A}$ with $A \subseteq B$. Suslin towers were studied in [5]. There, P. Borodulin-Nadzieja and D. Chodounský proved, in particular, that Suslin ω_1-pretowers exist under $\mathfrak{b} = \omega_1$. In fact, whenever $\mathcal{B}$ is an increasing family of functions in ω^ω with respect to $<^*$ of order type ω_1, then the family $\langle T_f \rangle_{f \in \mathcal{B}}$, where

$$T_f = \{(n, m) : m \leq f(n)\}$$

is an ω_1-pretower. Also notice that if $f, g \in \mathcal{B}$ are such that $f \leq g$, then $T_f \subseteq T_g$. Consequently, if $(\mathcal{B}, \leq)$ has no uncountable pies then $\langle T_f \rangle_{f \in \mathcal{B}}$ is Suslin. Thus, we have the following corollary.

Corollary 6.20 (CA_2)**.** *There is a Suslin ω_1-pretower.*

The previous corollary contradicts the Open Graph Axiom (OGA). As with the case of PID, OGA (also known as OCA, $OCA_{[T]}$ or Tordorcevic Axiom TA) is an axiom introduced by the third author in [83]. The reader may consult [18], [38], [46], [51], [73], [75] and [78] for more information about OGA.

6.3. **S-spaces.** The results contained in this subsection as well as their proofs are completely based in [83]. For that reason, most of the proofs will be omitted here.

Definition 6.21. *Let (X, τ) be a topological space with $|X| = \omega_1$ and $\langle x_\alpha \rangle_{\alpha \in \omega_1}$ be an enumeration of X. We say that:*

- *X is left-open (right-separated) if $\{x_\xi : \xi \leq \alpha\}$ is open for every $\alpha \in \omega_1$.*
- *X is an S-space if it is T_3, hereditarily separable and not Lindelöf.*
- *X is a strong S-space if all of its finite powers are S-spaces.*
- *X is right-open (left-separated) if $\{x_\xi : \xi \geq \alpha\}$ is open for every $\alpha \in \omega_1$.*
- *X is an L-space if it is T_3, hereditarily Lindelöf and not separable.*
- *X is a strong L-space if all of its finite powers are L-spaces.*

The existence of an S-space used to be one of the main open problems in set-theoretic topology. Such spaces exist under a large variety of axioms (like CH and some parametrized diamonds of [52]). This question was finally settled when the third author proved that the Proper Forcing Axiom (PFA) implies that there are no S-spaces. Here we will construct S-spaces using 2-capturing construction schemes. To learn more about S-spaces (and L-spaces) the reader may consult [1], [61], [60] and [83]. The following is known:

Lemma 6.22. *Let (X, τ) be a topological space with $|X| = \omega_1$ and $\langle x_\alpha \rangle_{\alpha \in \omega_1}$ be an enumeration of X. If:*

(*a*) *X is T_3,*
(*b*) *X is locally countable,*
(*c*) *X does not have uncountable discrete sets.*

Then (X, τ) is an S-space.

Corollary 6.23. *Let (X, τ) be a T_3 topological space and $\langle x_\alpha \rangle_{\alpha \in \omega_1}$ be an enumeration of X. If X is left-open and does not have uncountable discrete sets then it is is an S-space. Furthermore, if X^n does not have uncountable discrete sets for any $1 \leq n \in \omega$, then X is a strong S-space.*

Let o^* be a coloring satisfying the conclusion of Corollary 6.8. For each $\alpha \in \omega_1$ define $x_\alpha : \omega_1 \longrightarrow 2$ as follows:

$$x_\alpha(\beta) = \begin{cases} \min(o^*(\alpha, \beta), 1) & \text{if } \alpha < \beta \\ 0 & \text{if } \alpha > \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

Consider $X = \langle x_\alpha \rangle_{\alpha \in \omega_1}$ endowed with the product topology inherited by 2^{ω_1}. Then it is clear that X is T_3. Furthermore, for each $\alpha \in \omega_1$, the set $\{x_\beta : \beta \in \omega_1 \textit{ and } x_\beta(\alpha) = 1\}$ is an open set contained in $\{x_\xi \,|\, \xi \leq \alpha\}$ and having α as an element. Hence, X is also left-open. Making use of Lemma 6.23 and the Ramsey property associated to o^* it is easy to see that X is a strong S-space. In the same way, we can define for each $\alpha \in \omega_1$, y_α as follows:

$$y_\alpha(\beta) = \begin{cases} \min(o^*(\alpha, \beta), 1) & \text{if } \alpha > \beta \\ 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

By a similar argument, one can show that $\langle y_\alpha \rangle_{\alpha \in \omega_1}$ is a strong L-space. For more detail the reader can read the Chapter 2 of [83]. It is worth pointing out that unlike the case of

S-spaces, the L-spaces exist in ZFC (see [48]).

Now, we will build another S-space using our family $\mathcal{B}_{\mathcal{F}}$. Given $\alpha \in \omega_1$, let $C(\alpha) = \{f_\xi : f_\xi \leq f_\alpha\}$. We define τ_S to be the topology over $\mathcal{B}_{\mathcal{F}}$ obtained by refining the canonical Baire topology of ω^ω restricted to $\mathcal{B}_{\mathcal{F}}$ by declaring the sets $C(\alpha)$ open.

Remark 6.24. *It is straight forward that each $\alpha \in \omega_1$ has as a local base the following family:*

$$\{C(\alpha) \cap [s] : s \in \omega^{<\omega} \text{ and } f_\alpha \in [s]\}.$$

Here, $[s] = \{f \in \omega^\omega : s \subseteq f\}$ The following is based on Todorčević proof that $\mathfrak{b} = \omega_1$ implies that there is an S-space.

Proposition 6.25. *$(\mathcal{B}_{\mathcal{F}}, \tau_S)$ is an S-space.*

Proof. This proposition will be proved by appealing to the Lemma 6.23. For each $\alpha \in \omega_1$ we have that $C(\alpha)$ is closed in the Baire topology. Therefore, such set is clopen in τ_S. From this it easily follows that $\mathcal{B}_{\mathcal{F}}$ is 0-dimensional. Consequently, $\mathcal{B}_{\mathcal{F}}$ is also regular . Moreover, $\mathcal{B}_{\mathcal{F}}$ is left-open by definition. Thus, the only thing left to show is that $\mathcal{B}_{\mathcal{F}}$ does not contain any uncountable discrete set.

Let $S \in [\omega_1]^{\omega_1}$ and assume towards a contradiction that $\langle f_\alpha \rangle_{\alpha \in S}$ is discrete. In this way, for each $\alpha \in S$ we can find $s_\alpha \in \omega^{<\omega}$ so that, for $U_\alpha = C(\alpha) \cap [s_\alpha]$, we have that $U_\alpha \cap Y = \{\alpha\}$. Let $W \in [S]^{\omega_1}$ and $s \in \omega^{<\omega}$ for which $s_\alpha = s$ for all $\alpha \in W$. According to the Theorem 6.5, there are $\alpha < \beta \in W$ for which $osc(\alpha, \beta) = 0$. In other words, $f_\alpha < f_\beta$. By definition of $C(\beta)$, $f_\alpha \in C(\beta)$. Thus, $f_\alpha \in U_\beta$ which is a contradiction. □

Now, we present the construction of a distinct S-space. For this construction we adapt the ideas from Chapter 2 of [83], for that reason we will avoid most of the proofs.

Definition 6.26. *Given $\beta \in \omega_1$, we define $H(\beta) = \{\alpha < \beta : \rho(\alpha, \beta) = \Delta(\alpha, \beta)\}$.*

Given $\beta \in \omega_1$, the set of all $\alpha < \beta$ so that $\{\alpha, \beta\}$ is captured, is contained in $H(\beta)$. Furthermore, if $\mathcal{F}$ is of type $\langle m_k, 2, r_{k+1}\rangle_{k\in\omega}$, then these two sets are equal. As a consequence of this, and since $\mathcal{F}$ is 2-capturing, we also have the following.

Lemma 6.27. *Let $\mathcal{S} \subseteq FIN(\omega_1)$ be an uncountable family of pairwise disjoint sets. Then there are $a, b \in \mathcal{S}$ of the same cardinality n such that $a < b$ and $a(i) \in H(b(i))$ for all $i < n$.*

Definition 6.28. *For each $\beta \in \omega_1$, we recursively define $C(\beta) \subseteq \beta + 1$ as the set containing β and all $\alpha < \beta$ for which there is $\gamma \in H(\beta)$ such that:*

(a) *$\alpha \in C(\gamma)$,*
(b) *for all $\gamma \neq \xi \in H(\beta) \cup \{\beta\}$, $\Delta(\alpha, \gamma) > \Delta(\alpha, \xi)$.*

Finally, we define $C_k(\beta) = \{\alpha \in C(\beta) : \Delta(\alpha, \beta) \geq k\}$ for each $k \in \omega$.

Lemma 6.29. *Let $\beta \in \omega_1$ and $\gamma \in H(\beta)$. Then $C_l(\gamma) \subseteq C(\beta)$ for $l = \Delta(\gamma, \beta) + 1$.*

Lemma 6.30. *Let $\beta \in \omega_1$. For each $k \in \omega$ and $\alpha \in C_k(\beta)$ there is $l \in \omega$ such that $C_l(\alpha) \subseteq C(\beta)$.*

By the previous corollary it is easy to see that the set $\{C_k(\beta) : k \in \omega \text{ and } \beta \in \omega_1\}$ forms a base for a topology in ω_1. It turns out that this defines a first countable locally compact strong S-space. It is convenient to transfer such topology to the family $\mathcal{B}_{\mathcal{F}}$.

Definition 6.31 (The topology τ_C). *Let $\beta \in \omega_1$ and $k \in \omega$. We define $\hat{C}_k(\beta) = \{f_\alpha : \alpha \in C_k(\beta)\}$ and $\hat{C}(\beta) = \{f_\alpha : \alpha \in C(\beta)\}$. Note that $\{\hat{C}_k(\beta) : k \in \omega \text{ and } \beta \in \omega_1\}$ forms a base for a topology over $\mathcal{B}_\mathcal{F}$. We will call this topology τ_C.*

The following lemma follows directly from the fact that $\hat{C}_k(f_\beta) \subseteq [f_\beta|_k]$ for each $\beta \in \omega_1$ and $k \in \omega$.

Lemma 6.32. *Let $s \in \omega^{<\omega}$. Then $[s] \cap \mathcal{B}_\mathcal{F}$ is open in $(\mathcal{B}_\mathcal{F}, \tau_C)$. In particular, $(\mathcal{B}_\mathcal{F}, \tau_C)$ is Hausdorff.*

Lemma 6.33. *Let $\beta \in \omega_1$. $\hat{C}_k(\beta)$ is compact for each $k \in \omega$.*

Proposition 6.34. *$(\mathcal{B}_\mathcal{F}, \tau_C)$ is a locally compact strong S-space.*

Proof. By definition, $\mathcal{B}_\mathcal{F}$ is left-open and it is locally compact due to Lemma 6.33. By Lemma 6.32, $\mathcal{B}_\mathcal{F}$ is also Hausdorff. The last two properties imply that the space is T_3. Fix $n \in \omega$. It remains to prove that $\mathcal{B}_\mathcal{F}^{n+1}$ has no uncountable discrete subspaces. For this, let $S \subseteq \omega_1^n$ be uncountable, and assume towards a contradiction that $\langle (f_{x(0)}, \dots, f_{x(n)}) \rangle_{x \in S}$ is discrete. Without loss of generality we can suppose $x(i) < x(j)$ whenever $i < j \leq n$ and $\langle x(i) \rangle_{i \leq n} \cap \langle y(i) \rangle_{i \leq n} = \emptyset$ for all $x, y \in S$ with $x \neq y$. Furthermore, be a refining argument we can also suppose there is $k \in \omega$ such that for all $x, y \in S$, the following happens:

(1) $\big(\prod_{i \leq n} C_k(x(i))\big) \cap S = \{x\}$,
(2) $f_{x(i)}|_k = f_{y(i)}|_k$ for every $i \leq n$.

Due to Lemma 6.27, we know there are distinct $x, y \in S$ such that $x(i) \in H(y(i))$ for every $i \leq n$. For any such i, we know $\Delta(x(i), \xi) = \omega$ if and only if $\xi = x(i)$. Since $x(i)$ clearly belongs to $C(x(i))$, it follows from the definition that $x(i) \in C(y(i))$. But $f_{x(i)}|_k = f_y(i)|_k$, so $x(i)$ in fact is an element of $C_k(x(i))$. In this way, we conclude that $x \in \big(\prod_{i \leq n} C_k(y(i))\big) \cap S$, which is a contradiction. □

Proposition 6.35. *The Alexandroff compactification of $(\mathcal{B}_\mathcal{F}, \tau_C)$ is a compact strong S-space.*

Corollary 6.36 (CA_2). *There is a scattered compact strong S-space K whose function space $C(K)$ is hereditarily weakly Lindelöf and whose space $P(K)$ of all probability measures is also a strong S-space.*

6.4. **A second look into Baumgartner Axiom.** We do not know if CA_2 implies the existence of an entangled sets. However, in this subsection we will prove it does imply the negation of $BA(\omega_1)$. Our proof is based on the third author's proof that $\mathfrak{b} = \omega_1$ implies the failure of $BA(\omega)$ (see [69]). Remember that at the beginning of this section we fixed 2-capturing construction scheme, namely $\mathcal{F}$. Remember that at the beginning of this section we fixed 2-capturing construction scheme, namely $\mathcal{F}$.

Lemma 6.37. *For any $\alpha \in \omega_1$ there are infinitely many $k \in \omega$ for which $0 \leq \Xi_\alpha(k) < n_k - 1$.*

Proof. Let $\alpha \in \omega_1$. Assume towards a contradiction that there is $k \in \omega$ so that for any $l > k$ we have that $\Xi_\alpha(l) = n_l - 1$. Since ρ is an unbounded metric, there are $\beta < \gamma \in \omega_1$ so that $\alpha < \beta$ and $l = \rho(\beta, \gamma) > k$. According to the Lemma 3.31 we have that $\Xi_\beta(l) \leq \Xi_\gamma(l) \leq n_l - 1$.

Let $F \in \mathcal{F}_l$ be such that $\{\beta, \gamma\} \subseteq F$ and $a = \|\alpha\|_l$. Observe that since $\alpha < \beta$ and $\Xi_\beta(l) < n_l - 1$ then $F_{n_l-1} \backslash R(F) \subseteq \omega_1 \backslash (\alpha+1)$. In this way, $\alpha < F(a)$. This is because $\Xi_\alpha(l) = n_l - 1$. To finish, just note that $\|\alpha\|_l = \|F(a)\|_l$. This means that

$$\rho(\alpha, F(a)) \geq \Delta(\alpha, F(a)) \geq l > k.$$

By the point (b) of Lemma 3.31 we get that $0 \leq \Xi_\alpha(l') < \Xi_{F(a)}(l')$ where $l' = \rho(\alpha, F(a))$. This is a contradiction, so the proof is over. $\square$

Now, let us fix M a countable elementary submodel of some large enough $H(\theta)$ with $\mathcal{F}, \mathcal{B}_\mathcal{F} \in M$. Define $\mathcal{A} = \mathcal{B}_\mathcal{F} \backslash M$. The following lemma is a direct consequence of elementarity.

Lemma 6.38. *Let $s \in \omega^{<\omega}$. If $\mathcal{A} \cap [s] \neq \emptyset$, then $\mathcal{A} \cap [s]$ is uncountable.*

Remember that we can think of $(\mathcal{A}, <_{lex})$ (The lexicographical order) as a subset of $\mathbb{R}$.

Lemma 6.39. *$(\mathcal{A}, <_{lex})$ is ω_1-dense.*

Proof. Let $f_\alpha, f_\beta \in \mathcal{A}$ with $f_\alpha <_{lex} f_\beta$, and let $l = \Delta(\alpha, \beta)$. According to the Lemma 6.37, there is $k > l$ so that $0 \leq \Xi_k(\alpha) < n_k - 1$. Consider $F \in \mathcal{F}_k$ such that $\alpha \in F$. Then $\alpha \in F \backslash F_{n_k-1}$. Thus, the unique $\gamma \in F_{n_k-1}$ for which $\|\gamma\|_{k-1} = \|\alpha\|_{k-1}$ is greater than α. Furthermore, $\|\gamma\|_k > \|\alpha\|_k$ and $f_\gamma \in \mathcal{A}$ because $f_\alpha \in \mathcal{A}$. In this way, $S = \mathcal{A} \cap [f_\gamma|_k]$ is uncountable due to the Lemma 6.38. In order to finish, we will show that S is contained in the open interval given by f_α and f_β. Indeed, let $f_\xi \in S$. By definition, $f_\xi|_k = f_\alpha|_k$. This implies that $f_\xi <_{lex} f_\beta$ because $k > l$. Finally, since $f_\alpha(k) = \|\alpha\|_k < \|\gamma\|_k = f_\gamma(k) = f_\xi(k)$ then $f_\alpha <_{lex} f_\xi$. $\square$

Definition 6.40. *Let $\alpha \in \omega_1$. We define $h_\alpha : \omega \longrightarrow \omega$ as:*

$$h_\alpha(i) = m_i - f_\alpha(i.)$$

Additionally, we let $-\mathcal{A} = \{h_\alpha(i) : \alpha \in \omega_1 \backslash M\}$.

It is easy to see that $f_\alpha <_{lex} f_\beta$ if and only if $h_\beta <_{lex} h_\alpha$. Hence, we have the following corollary.

Corollary 6.41. *$(-\mathcal{A}, <_{lex})$ is ω_1-dense.*

Proposition 6.42. *There is no increasing function from $\mathcal{A}$ to $-\mathcal{A}$.*

Proof. Let us assume towards a contradiction that there is an increasing function $\Psi : \mathcal{A} \longrightarrow -\mathcal{A}$ increasing. Let $\psi : \omega_1 \backslash M \longrightarrow \omega_1 \backslash M$ be the unique function so that $\Psi(f_\alpha) = h_{\psi(\alpha)}$.

<u>Claim</u>: ψ has at most on fixed point.

Proof. Suppose that this is not true and let $\alpha, \beta \in \omega_1$ be distinct fixed points of ψ such that $f_\alpha <_{lex} f_\beta$. As Ψ is increasing, then

$$h_\alpha = h_{\psi(\alpha)} = \Psi(f_\alpha) <_{lex} \Psi(f_\beta) = h_{\psi(\beta)} = h_\beta.$$

But this is a contradiction since, in fact, $h_\beta <_{lex} h_\alpha$. $\square$

Now, let $X = \{\alpha \in \omega_1 : \alpha < \psi(\alpha)\}$ and $Y = \{\alpha \in \omega_1 : \alpha > \psi(\alpha)\}$. By the previous claim, one of this sets is uncountable. Suppose without loss of generality that X is uncountable. For each $\alpha \in X$, let $b_\alpha = \{\alpha, \phi(\alpha)\}$. Since $\mathcal{F}$ is 2-capturing, we can find distinct $\alpha, \beta \in X$ for which $\{b_\alpha, b_\beta\}$ is captured. Observe that $f_\alpha <_{lex} f_\beta$ and $f_{\psi(\alpha)} <_{lex} f_{\psi(\beta)}$, or equivalently, $\Psi(f_\alpha) = h_{\psi(\alpha)} >_{lex} h_{\psi(\beta)} = \Psi(f_\beta)$. Note that this is a contradiction. Thus, the proof is over. $\square$

Corollary 6.43 (CA_2)**.** *There are two ω_1-dense sets of reals which are not isomorphic.*

7. Capturing schemes from $\diamondsuit$

In [77] there is a proof that $\diamondsuit$-principle implies FCA. Unfortunately, the proof presented there is incomplete. Fortunately, the Theorem is true. We present a correct proof in this section. As we have done in the previous sections, we also fix a type $\langle m_k, n_{k+1}, r_{k+1}\rangle_{k\in\omega}$.

Definition 7.1 (The restriction of a scheme)**.** *Let $\mathcal{F}$ be a construction scheme over X. For any $Y \subseteq X$, we define the restriction of $\mathcal{F}$ to Y as*

$$\mathcal{F}|_Y = \{F \in \mathcal{F} : F \subseteq Y\}.$$

The following remarks will be useful and we will use them without any explicit mention.

Remark 7.2. *The following properties hold for each construction scheme $\mathcal{F}$:*

- *If $Y \in \mathcal{F}$, then $\mathcal{F}|_Y$ is a construction scheme over Y.*
- *If $Y \subseteq X$ and $k \in \omega$, then $(\mathcal{F}|_Y)_k = \{F \in \mathcal{F}_k : F \subseteq Y\}$.*
- *If $\mathcal{F}$ is a construction scheme over a finite set X, then $X \in \mathcal{F}$.*

Remark 7.3. *If $\mathcal{F}$ is a construction scheme over X, and $Y \subseteq X$ is such that $\mathcal{F}|_Y$ is a construction scheme over Y, then $\rho_{\mathcal{F}}|_{Y^2} = \rho_{\mathcal{F}|_Y}$.*

The next proposition is easily proved by recursion. Its proof already appear in [77] and [42]. For that reason we will not include it here.

Proposition 7.4 ([77])**.** *Let X be a finite set of ordinals. Then there is a construction scheme over X if and only if $|X| = m_k$ for some $k \in \omega$. Furthermore, if this situation occurs, then such construction scheme is unique. Thus, we will call it $\mathcal{F}(X)$.*

Remark 7.5. *If $\mathcal{F}$ is a construction scheme over X and $F \in \mathcal{F}$, then $\mathcal{F}|_F = \mathcal{F}(F)$. Also, if X and Y are finite sets of cardinality m_k and $\phi : X \longrightarrow Y$ is the increasing bijection, then $\mathcal{F}(Y) = \{\, \phi[F] : F \in \mathcal{F}(X)\}$.*

We now present some fundamental results needed in order to recursively define construction schemes over ω_1. All of them appear (at least implicitly) in [77].

Lemma 7.6 ([77])**.** *Suppose that X and Y are two sets of ordinals for which $X \subseteq Y$. Let $\mathcal{F}$ and $\mathcal{G}$ be construction schemes over X and Y respectively. If $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}|_X$.*

Lemma 7.7 ([77])**.** *Let $\mathcal{B}$ be a family of sets of ordinals which is totally ordered with respect to $\subseteq$. Suppose that $\langle \mathcal{F}^X\rangle_{X\in\mathcal{B}}$ is a sequence so that for any $X, Y \in \mathcal{B}$ with $X \subseteq Y$, the following conditions hold:*

- *$\mathcal{F}^X$ is a construction scheme over X,*
- *$\mathcal{F}^Y|_X = \mathcal{F}^X$ (or equivalently, $\mathcal{F}^X \subseteq \mathcal{F}^Y$).*

Then $\mathcal{F} = \bigcup\limits_{X\in\mathcal{B}} \mathcal{F}^X$ is a construction scheme over $Z = \bigcup \mathcal{B}$.

Lemma 7.8 ([77])**.** *Let X be a set of ordinals, $\mathcal{F}$ be a construction scheme over X and $k < l \in \omega$. If $F \in \mathcal{F}_l$ and $\alpha \in F$, then there is (a unique) $E \in \mathcal{F}_k$ such that:*

(a) *$E \subseteq F$,*
(b) *$\alpha \in E$,*
(c) *$E\backslash\alpha$ is an interval in F.*

Corollary 7.9. *Let $\mathcal{F}$ be a construction scheme over X. Suppose that $\alpha \in X$ and $k \in \omega$. Then there is (a unique) $E \in \mathcal{F}_k$ such that $\alpha \in E$ and $E \backslash \alpha$ is an interval in X.*

Remark 7.10. *If we apply the previous corollary to the case where X is an ordinal, say γ, we get the following: For any $\alpha \in \gamma$ and $k \in \omega$, we have that*

$$(\alpha)_k^- \cup [\alpha, \alpha + (m_k - \|\alpha\|_k)\,)$$

is an element of $\mathcal{F}_k$.

The existence part of the proposition mention below already appears in [77]. For the sake of completeness, we believe that it is worth proving the uniqueness part.

Proposition 7.11 ([77])**.** *There is a unique construction scheme over ω. We will call it $\mathcal{F}(\omega)$.*

Proof. As we said before, we are only proving the uniqueness. Suppose that $\mathcal{F}'$ is another construction scheme over ω. Let $k \in \omega$. According to the Remark 7.10, the set

$$(0)_k^- \cup [0, 0 - (m_k - \|0\|_k)\,) = m_k$$

is an element of $\mathcal{F}'$. Thus, $\mathcal{F}(m_k) = \mathcal{F}'|_{m_k} \subseteq \mathcal{F}'$. Since this is true for any k, we have that $\mathcal{F} \subseteq \mathcal{F}'$. But then $\mathcal{F} = \mathcal{F}'$ due to Lemma 7.6. □

The metric structure of closed sets is fully determined by their cardinality (see Lemma 3.18). In particular, for each $\beta \in \gamma$ and $l \in \omega$, the set $(\beta)_l$ is ρ-isomorphic to $\|\beta\|_l + 1 = [0, \|\beta\|_l]$ (recall $\|\beta\|_l = |(\beta)_l^-|$). It will be convenient to name and explicitly describe the increasing bijections between these two sets.

Definition 7.12. *Let $\mathcal{F}$ be a construction scheme over a limit ordinal γ. Given $\beta \in \gamma$ and $l \in \omega$, we define $\phi_l^\beta : (\beta)_l \longrightarrow \|\beta\|_l + 1$ as:*

$$\phi_l^\beta(\alpha) = \|\alpha\|_l.$$

The inverse function of ϕ_l^β is given by:

$$(\phi_l^\beta)^{-1}(i) = (\beta)_l(i).$$

Proposition 7.13. *Let $\mathcal{F}$ be a construction scheme over an ordinal γ. Given $\alpha, \beta \in \gamma$ and $k \le l \in \omega$, the following conditions hold:*

(*a*) $\|\beta\|_k = \|\,\|\beta\|_l\,\|_k$,
(*b*) *If $l \ge \rho(\alpha, \beta)$, then $\rho(\alpha, \beta) = \rho(\|\alpha\|_l, \|\beta\|_l)$*
(*c*) *If $l \ge \Delta(\alpha, \beta)$, then $\Delta(\alpha, \beta) = \Delta(\|\alpha\|_l, \|\beta\|_l)$.*

We can rewrite the point (a) of the previous proposition in a more useful way.

Lemma 7.14. *Let $\mathcal{F}$ be a construction scheme over an ordinal γ. Given $\beta \in \gamma$ and $k \le l \in \omega$, we have that*

$$\phi_k^{\|\beta\|_l} \circ \phi_l^\beta = \phi_k^\beta.$$

7.1. **The forcing $\mathbb{P}(\mathcal{F})$ and schemes in ZFC.** In this subsection we will present an incarnation of the Cohen forcing which was first considered in [77]. This forcing will serve the purpose of extending a construction scheme from a countable limit ordinal to the next one, and will play a key role in the arguments of the next subsections of this section.
Let us fix a construction scheme $\mathcal{F}$ of type τ over a limit ordinal γ.

Definition 7.15 (The forcing $\mathbb{P}(\mathcal{F})$)**.** *We define $\mathbb{P}(\mathcal{F})$ as the forcing consisting of the empty set and of all $p \in FIN(\gamma+\omega)$ with the following properties:*

(I) There is $k_p \in \omega$ such that $|p| = m_{k_p}$.
(II) There is $F \in \mathcal{F}_{k_p}$ such that $p \cap \gamma \sqsubseteq F$.
(III) $p \cap [\gamma, \gamma+\omega)$ is an initial segment of $[\gamma, \gamma+\omega)$.

Whenever $p \cap \gamma \neq \emptyset$ (even if p is not a condition of $\mathbb{P}(\mathcal{F})$), we let $\alpha_p = \max(p \cap \gamma)$. Additionally, for each $k \in \omega$ we let $\mathbb{P}_k(\mathcal{F}) = \{\, p \in \mathbb{P}(\mathcal{F}) : k_p = k \,\}$. The order on $\mathbb{P}(\mathcal{F})$ is given by

$$p \leq q \text{ if and only if } q \in \mathcal{F}(p) \text{ or } q = \emptyset.$$

Note that $\mathbb{P}(\mathcal{F})$ is always countable. Therefore, it is forcing equivalent to the Cohen forcing.

Remark 7.16. *From Lemma 3.9 it follows that the condition (II) of Definition 7.15 is equivalent to:*

$$\text{(II)}^* \;\; p \cap \gamma = (\alpha_p)_{k_p},$$

in the case where $p \cap \gamma \neq \emptyset$.

In general, if $p \in \mathbb{P}(\mathcal{F})$, there are many elements $F \in \mathcal{F}_{k_p}$ testifying the condition (II) of Definition 7.15 with respect to p. However, it turns out that there is a canonical one. Such F will be useful in many of the arguments involving the forcing $\mathbb{P}(\mathcal{F})$. In the following definition, we explicitly describe it.

Definition 7.17. *If $p \in FIN(\gamma+\omega)$ and $\delta \leq \gamma$, we define the reduction of p to δ as follows:*

$$red_\delta(p) = \begin{cases} (p \cap \delta) \cup [\max(p\cap\delta)+1, \max(p \cap \delta) + |p \backslash \delta| + 1) & \text{if } p \cap \delta \neq \emptyset \\ |p| & \text{if } p \cap \delta = \emptyset \end{cases}$$

Remark 7.18. *The reduction operation is closely related to the Corollary 7.9. Suppose that we take $\alpha \in \gamma$ and $k \in \omega$. According to the corollary, we know that there is a unique $E \in \mathcal{F}_k$ so that $\alpha \in E$ and $E\backslash\alpha$ is an interval in γ. It turns out that such E can be described using the reduction operation. In order to do this, take an arbitrary $F \in \mathcal{F}_k$ with $\alpha \in F$. Then:*

$$\begin{aligned} red_{\alpha+1}(F) &= F \cap (\alpha+1) \cup [\, \alpha+1,\, \alpha+1+|F\backslash(\alpha+1)|\,) \\ &= (\alpha)_k^- \cup [\, \alpha,\, \alpha+1+(m_k - (\|\alpha\|_k + 1))\,) \\ &= (\alpha)_k^- \cup [\, \alpha,\, \alpha+m_k - \|\alpha\|\,). \end{aligned}$$

Due to the Remark 7.10, it is straightforward that $E = red_{\alpha+1}(F)$.

The reduction operation will be mainly used in the case where $p \in \mathbb{P}(\mathcal{F})$ and $\alpha = \gamma$. By arguing in a similar manner as in the Remark 7.18, we get the following lemma. The proof of it is left to the reader.

Lemma 7.19. *If $p \in FIN(\gamma+\omega)$ is such that $p \cap [\gamma, \gamma+\omega)$ is an initial segment of $[\gamma, \gamma+\omega)$, then the following conditions are equivalent:*

(a) $p \in \mathbb{P}(\mathcal{F})$,
(b) $red_\gamma(p) \in \mathcal{F}$.

Even though the previous lemma is useful, the main tool for defining and extending conditions in $\mathbb{P}(\mathcal{F})$ relies in the next definition.

Definition 7.20. *Let $F \in FIN(\gamma)$ and $\alpha \in \gamma$. We define the cut of F at α as follows:*

$$Cut_\alpha(F) = (F \cap \alpha) \cup [\gamma, \gamma + |F \backslash \alpha|)$$

Remark 7.21. *Suppose that $F \in \mathcal{F}$ and $\alpha \in \gamma$. If $\phi : F \longrightarrow Cut_\alpha(F)$ is the increasing bijection, then*

$$\mathcal{F}(Cut_\alpha(F)) = \{\, \phi[H] \,:\, H \in \mathcal{F}(F)\,\}$$

due to the Remark 7.5. In particular, this means that $Cut_\alpha(F_i)$ is the ith element of the canonical decomposition of $Cut_\alpha(F)$ for each $i < n_{\rho^F}$. In other words, $Cut_\alpha(F_i) = (Cut_\alpha(F))_i$. On the other hand, it is not necessarily true that $\phi[H] = Cut_\alpha(H)$ for a given $H \in \mathcal{F}(F)$. A particular case in which the previous equality holds is when $\alpha \in H$ and $H\backslash\alpha$ is an interval in F (Thus, an initial segment of $F\backslash\alpha$).

Remark 7.22. *The reduction and cut operations are in some sense inverses of each other. That is, if $p \in \mathbb{P}(\mathcal{F})$ and $p \cap \gamma \neq \emptyset$, then $Cut_{\alpha_p+1}(red_\gamma(p)) = p$. On the other hand, if $F \in FIN(\gamma)$ and $\alpha \in F$ then $red_\gamma(Cut_{\alpha+1}(F)) = F$.*

Lemma 7.23. *Let $F \in \mathcal{F}$ and $\alpha \in \gamma$. Then $Cut_\alpha(F) \in \mathbb{P}(\mathcal{F})$. Furthermore, if $F, G \in \mathcal{F}$ are such that $F \subseteq G$ and $\alpha \in F$ then $Cut_\alpha(G) \leq Cut_\alpha(F)$.*

Proof. For the first part of the lemma, suppose that $F \in \mathcal{F}$ and $\alpha \in \gamma$. Let $k = \rho^F$. Then $m_k = |F| = |Cut_\alpha(F)|$. In this way, $Cut_\alpha(F)$ satisfies the condition (I) of Definition 7.15. Now, $Cut_\alpha(F) \cap \gamma = F \cap \alpha \sqsubseteq F$. Therefore, the condition (II) of the same definition holds for $Cut_\alpha(F)$. Finally, $Cut_\alpha(F) \cap [\,\gamma,\, \gamma + \omega\,) = [\,\gamma,\, \gamma + |F\backslash\alpha|\,)$. Thus, $Cut_\alpha(F)$ satisfies the condition (III) of Definition 7.15, which means that $Cut_\alpha(F) \in \mathbb{P}(\mathcal{F})$.

Next, we prove the second part of the lemma. Let $F, G \in \mathcal{F}$ be such that $F \subseteq G$ and let $\alpha \in F$. If $F = G$, the result is obvious. So we may assume that the inclusion between F and G is proper. It follows that $k < l$ where $k = \rho^F$ and $l = \rho^G$. Therefore, we are in the conditions of applying the lemma 7.8 to α, G and k. In this way, we get $E \in \mathcal{F}_k$ for which $\alpha \in E$, $E \subseteq G$ and $E\backslash\alpha$ is an interval in G. Note that $|E\backslash\alpha| = |F\backslash\alpha|$ and $F \cap \alpha = (\alpha)^-_k = E \cap \alpha$. Thus, $Cut_\alpha(F) = Cut_\alpha(E)$. Now, let $\phi : G \longrightarrow Cut_\alpha(G)$ be the increasing bijection. Due to the Remark 7.21,

$$\mathcal{F}(Cut_\alpha(G)) = \{\, \phi[H] \,:\, H \in \mathcal{F}(G)\,\}.$$

Finally, just note that $\phi[E] = Cut_\alpha(E)$ by virtue of the same remark. $\square$

Lemma 7.24. *Let $k \in \omega$, $p \in \mathbb{P}_k(\mathcal{F})$ (that is, $k = k_p$) and $\alpha \in \gamma$ be such that $(\alpha)^-_k = p \cap \gamma$. Then $Cut_\alpha(G) \leq p$ for each $G \in \mathcal{F}$ with $\alpha \in G$ and $\rho^G \geq k$.*

Proof. Let $G \in \mathcal{F}$ be as in the hypotheses. Consider $F \in \mathcal{F}_k$ for which $\alpha \in F$ and $F \subseteq G$. According to the Lemma 7.23, $Cut_\alpha(G) \leq Cut_\alpha(F)$. To finish, just note that $|Cut_\alpha(F)| = |p|$ and $Cut_\alpha(G) \cap \gamma = G \cap \alpha = (\alpha)^-_k = p \cap \gamma$. In this way, $Cut_\alpha(F) = p$ due to the condition (III) of Definition 7.15.

$\square$

7.2. **Construction schemes in ZFC.** The next definition appeared for the first time in [77]. This hypothesis is sufficient to build construction schemes in a recursive manner.

Definition 7.25 ([77]). *Let $A \in FIN(\gamma)$, $\alpha \in \gamma$ and $F \in \mathcal{F}$. We say that $IH_1(\alpha, A, F)$ holds if:*

(1) $A \subseteq F_0$,
(2) $R(F) = F \cap \alpha$.

Additionally, we say that $\mathcal{F}$ satisfies IH_1 if for all $A \in FIN(\gamma)$ and $\alpha \in \gamma$, there is $F \in \mathcal{F}$ for which $IH_1(\alpha, A, F)$ holds.

Without exception, all the results appearing in this subsection were already presented in [77].

Proposition 7.26 ([77]). *Suppose that τ is a good type. Then $\mathcal{F}(\omega)$ satisfies IH_1.*

Proof. Let $A \in \mathrm{FIN}(\gamma)$ and $\alpha \in \omega$. Since τ is a good type, there is $k > \max(A)$ for which $r_{k+1} = \alpha$. Let $F = m_{k+1}$. Then $F \in \mathcal{F}(\omega)$, $F_0 = m_k \supseteq A$ and $R(F) = r_{k+1} = \alpha$. This finishes the proof. $\square$

The following lemma is easy.

Lemma 7.27 ([77]). *Suppose that γ is a limit of limit ordinals and $\mathcal{F}|_\delta$ satisfies IH_1 for each limit $\delta < \gamma$. Then $\mathcal{F}$ also satisfies IH_1.*

Definition 7.28. *Given a filter $\mathcal{G}$ over $\mathbb{P}(\mathcal{F})$, we define $\mathcal{F}^{\mathcal{G}}$ as $\bigcup_{p\in\mathcal{G}} \mathcal{F}(p)$. Finally, $\mathcal{F}^{Gen}$ denotes the name for $\mathcal{F}^{\mathcal{G}}$ where G is a generic filter.*

Lemma 7.29 ([77]). *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. Then the set $\{\, p \in \mathbb{P}(\mathcal{F}) \,:\, p \cap \gamma \neq \emptyset \,\}$ is open dense in $\mathbb{P}(\mathcal{F})$.*

Lemma 7.30 ([77]). *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. Then the set $\{\, p \in \mathbb{P}(\mathcal{F}) \,:\, p \cap [\gamma, \gamma + \omega) \neq \emptyset \,\}$ is open dense in $\mathbb{P}(\mathcal{F})$.*

Proof. Let $q \in \mathbb{P}(\mathcal{F})$. Without loss of generality we may assume that $q \neq \emptyset$. We need to find $p \leq q$ with $p \backslash \gamma \neq \emptyset$. If $q \backslash \gamma \neq \emptyset$, there is nothing to do. In this way, we may assume that $q \subseteq \gamma$. In this case, it follows that $q \in \mathcal{F}$. Let $\alpha = \alpha_p + 1 = \max(q) + 1$. Since $\mathcal{F}$ satisfies IH_1, there is $F \in \mathcal{F}$ with $q \cup \{\alpha\} \subseteq F_0$ and $\alpha \cap F = R(F)$. Note that q is actually a subset of $R(F)$. Let $p = Cut_\alpha(F)$. According to the Lemma 7.23, $p = Cut_\alpha(F) \leq Cut_\alpha(q) = q$. Furthermore, $p \backslash \gamma \neq \emptyset$. This finishes the proof. $\square$

Lemma 7.31 ([77]). *Suppose that $\mathcal{F}$ is a construction scheme over γ satisfying IH_1. For any $\alpha \in \gamma + \omega$, the set $\mathcal{D}'_\alpha = \{p \in \mathbb{P}(\mathcal{F}) \,:\, \alpha \in p\}$ is open dense in $\mathbb{P}(\mathcal{F})$.*

Lemma 7.32 ([77]). *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. For any $F \in \mathcal{F}$, the set $\mathcal{D}_F = \{\, p \in \mathbb{P}(\mathcal{F}) \,:\, F \in \mathcal{F}(p) \,\}$ is open dense in $\mathbb{P}(\mathcal{F})$.*

Lemma 7.33 ([77]). *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. For any $A \in FIN(\gamma + \omega)$ and each $\alpha \in \gamma + \omega$, the set*

$$\mathcal{E}_{\alpha,A} = \{\, p \in \mathbb{P}(\mathcal{F}) \,:\, \alpha \in p,\ A \subseteq p_0,\ \text{and } \alpha \cap p = R(p) \,\}$$

is dense in $\mathbb{P}(\mathcal{F})$.

Proposition 7.34 ([77]). *Suppose that τ is a good type and $\mathcal{F}$ satisfies IH_1. Let $\mathcal{G}$ be a filter over $\mathbb{P}(\mathcal{F})$ intersecting $\mathcal{D}'_\alpha$, $\mathcal{D}_F$ and $\mathcal{E}_{\alpha,A}$ for all $\alpha \in \gamma+\omega$, $A \in FIN(\gamma+\omega)$ and $F \in \mathcal{F}$[14]. Then $\mathcal{F}^{\mathcal{G}}$ is a construction scheme over $\gamma+\omega$ which contains $\mathcal{F}$ and satisfies IH_1.*

Suppose that τ is a good type. By means of the results 7.26, 7.34, 7.7, and 7.27, we can recursively construct, for each limit $\gamma < \omega_1$, a construction scheme $\mathcal{F}^\gamma$ which satisfies IH_1 and such that $\mathcal{F}^\delta \subseteq \mathcal{F}^\gamma$ whenever $\delta < \gamma$. If we use one more time the Lemmas 7.7 and 7.27, we can conclude that

$$\mathcal{F} = \bigcup_{\gamma \in Lim} \mathcal{F}^\gamma$$

is a construction scheme over ω_1 which satisfies IH_1. This proves Theorem 2.3.

7.3. **Diamond principle and FCA.** The purpose of this subsection is to prove that Jensen's $\diamondsuit$-principle implies FCA.[15] In order to do this, we will work in the same manner as we did in the Subsection 7.2. That is, we want to find a suitable property IH_2 which is satisfied by $\mathcal{F}(\omega)$. Furthermore, given a construction scheme $\mathcal{F}$ over a limit ordinal $\gamma \leq \omega_1$, we want the two following things to happen:

(a) If γ is a limit of limit ordinals and $\mathcal{F}|_\delta$ satisfies IH_2 for each limit $\delta < \gamma$, then $\mathcal{F}$ also satisfies IH_2.

(b) If γ is countable and $\mathcal{F}$ satisfies IH_1 and IH_2 then there is a construction scheme $\mathcal{F}'$ over $\gamma+\omega$ containing $\mathcal{F}$ which satisfies IH_1 and IH_2.

By doing this, we may conclude that there is a construction scheme over ω_1 which satisfies IH_2. Finally, we want that:

(c) Any construction scheme over ω_1 which satisfies IH_2 is fully capturing.

While the points (a) and (c) are relatively easy to guarantee, it turns out that finding a property IH_2 which also satisfies the point (b) is a highly non-trivial problem. Unfortunately, the elegant property IH_2 defined in [77] is inconsistent, even though it satisfies (c).

For the rest of this section, we fix a countable limit ordinal γ and a construction scheme $\mathcal{F}$ be over γ of type τ. Moreover, we fix a $\diamondsuit$-sequence $\langle D_\alpha \rangle_{\alpha \in \mathrm{Lim}}$.

Definition 7.35 (Block interval sequence). *Let $j \in \omega$ and A be a non-empty subset of ω_1. We say that a sequence $\mathbb{I} = \langle \mathbb{I}(i) \rangle_{i<j} \subseteq FIN(A)$ is a block interval sequence over A if:*

- *For any $i < t$, $\mathbb{I}(i)$ is an interval in A.*
- *For all $i < i' < t$, $\mathbb{I}(i) < \mathbb{I}(i')$.*

Given $\alpha \in \omega_1$, we let $Bl(\alpha, A)$ to be the set of all interval sequences over $A \backslash \alpha$.[16] Given $\alpha \in X$, we denote $Bl(\,\|\alpha\|_k, m_k)$ simply as $Bl_k(\alpha)$ for each $k \in \omega$.

Remark 7.36. *Note that a block interval sequence $\mathbb{I}$ may be empty even though its elements are not. This case is not pathological, and will be highly relevant for the proof of Theorem 7.50.*

[14]Such $\mathcal{G}$ exists due to the Rasiowa-Sikorski Lemma

[15]At the end of this section, we will briefly how to modify the results presented here in order to show that $\diamondsuit$ actually implies $FCA(part)$. We do not prove this directly in order to simplify notation.

[16]Note that $Bl(\alpha, A) = Bl(0, A\backslash\alpha)$ for all $\alpha \in \omega_1$ and $A \in \mathrm{FIN}(\omega_1)$.

Definition 7.37. *Suppose that A and B are non-empty subsets of ω_1 and $\phi : A \longrightarrow B$ is an increasing function. Given $\mathbb{I} = \langle \mathbb{I}(i) \rangle_{i<j} \in Bl(0, A)$, we let*

$$\phi \circ \mathbb{I} = \langle\, \phi[\mathbb{I}(i)]\, \rangle_{i<j}.$$

Recall that if B has size m for some $m \in \omega$, then we identity the increasing bijection $\phi : m \longrightarrow B$ with B, and we denote the inverse of such function as B^{-1}. Following this convention, if $A = m$ and $\mathbb{I}$ is as before, $\phi\circ\mathbb{I}$ is denoted as $B\circ\mathbb{I}$. Analogously, if $\mathbb{J} \in Bl(0, B)$, we denote $\phi^{-1} \circ \mathbb{J}$ as $B^{-1} \circ \mathbb{J}$.

Remark 7.38. *Suppose that A, B, ϕ and $\mathbb{I}$ are as in the previous definition. Since ϕ is increasing, then $\phi \circ \mathbb{I}(i) < \phi \circ \mathbb{I}(i')$ for all $i < i < j$. However, it is not necessarily true that $\phi \circ \mathbb{I}(i)$ is an interval for a given i. A particular case in which this occurs is when $im(\phi)$ is itself an interval in B. In fact, it is easy to show that $im(\phi)$ is an interval in B if and only if*

$$\{\phi \circ \mathbb{I} \,:\, \mathbb{I} \in Bl(0, A)\} \subseteq Bl(0, B).$$

Moreover, if ϕ is surjective then the equality between the previous sets will hold. Particularly, if $\alpha \in \gamma$ and $k \in \omega$, then

$$Bl(\alpha, F) = \{\, F \circ \mathbb{I} \,:\, \mathbb{I} \in Bl_k(\alpha)\,\},$$
$$Bl_k(\alpha) = \{F^{-1} \circ \mathbb{I} \,:\, \mathbb{I} \in Bl(\alpha, F)\,\}$$

for all $F \in \mathcal{F}_k$ with $\alpha \in F$.

Definition 7.39 (Good pairs and good sets)**.** *Let $\alpha \in \omega_1$ and $A \in FIN(\omega_1)$. we say that a pair $(\mathbb{I}, z)$ is (α, A)-good if the following conditions hold:*

- *$z \in A\backslash\alpha$,*
- *$\mathbb{I} \in Bl(\alpha, A \cap (z + 1))$. That is, $\mathbb{I} \in Bl(\alpha, A)$ and either $\mathbb{I} = \emptyset$ or $\max(\bigcup \mathbb{I}) \leq z$.*

A non-empty set of ordered pairs T is said to be (α, A)-Good if every $(\mathbb{I}, z) \in T$ is (α, A)-good. We denote the family of (α, A)-Good sets as $Good(\alpha, A)$. Given $\alpha \in \gamma$ and $k \in \omega$, we denote $Good(\|\alpha\|_k, m_k)$ as $Good_k(\alpha)$.

Definition 7.40. *Suppose that A and B are non-empty subsets of ω_1 and $\phi : A \longrightarrow B$ is an increasing function. Given $\alpha \in \omega_1$ and $T \in Good(\alpha, A)$ we define*

$$\phi \bullet T = \{\, (\phi \circ \mathbb{I}, \phi(z)) \,:\, (\mathbb{I}, z) \in T\,\}.$$

In the particular case where $|B| = m$ for some $m \in \omega$ and $A = m$, we denote $\phi \bullet T$ as $B \bullet T$. Analogously, if $T' \in Good(\alpha, B)$ then $\phi^{-1} \bullet T'$ is denoted as $B^{-1} \bullet T'$.

Remark 7.41. *If $\alpha \in \gamma$ and $k \in \omega$ then*

$$Good(\alpha, F) = \{F \bullet T \,:\, T \in Good_k(\alpha)\,\},$$
$$Good_k(\alpha) = \{F^{-1} \bullet T \,:\, T \in Good(\alpha, F)\,\}$$

for each $F \in \mathcal{F}_k$ with $\alpha \in F$.

For the next the definition, it is convenient to recall the function ϕ^{β}_{l} that was presented in Definition 7.12.

Definition 7.42. *Let $\beta \in \gamma$ and $k \leq l \in \omega$. We define $\pi^{k,l}_{\beta} : \|\alpha\|_k + 1 \longrightarrow \|\beta\|_l + 1$ as:*

$$\pi^{k,l}_{\beta} = \phi^{\beta}_{l} \circ (\phi^{\beta}_{k})^{-1}.$$

It is not hard to see that $\pi^{k,l}_{\alpha}$ is just the increasing bijection from $\|\alpha\|_k + 1$ to $(\|\alpha\|_l)_k$.

Definition 7.43 (The $\bigstar$ relation)**.** *Let $\beta, \xi \in \gamma$, $2 \leq k < l \in \omega$, and $T \in Good_k(\beta)$. We say that T guesses (β, ξ, k, l), and write it as $T\bigstar(\beta, \xi, k, l)$, if there is $(\mathbb{I}, z) \in T$ for which:*

(a) $\|\beta\|_l \leq \|\xi\|_l$ *and* $\|\xi\|_k = z$,

(b) $(\xi)_k \circ \mathbb{I} \in Bl(0, (\xi)_l)$. *Equivalently, if* $\pi^{k,l}_\xi \circ \mathbb{I} \in Bl(0, m_l)$,

(c) $(\xi)_l(\|\beta\|_l) \in (\xi)_k$. *Equivalently,* $\|\beta\|_l \in (\|\xi\|_l)_k$.

Remark 7.44. *If $T\bigstar(\beta, \xi, k, l)$ and ξ' is such that $\|\xi'\|_l = \|\xi\|_l$ then $T\bigstar(\beta, \xi', k, l)$.*

Remark 7.45. *The previous definition will often be applied in cases where $\xi < \beta$.*

Recall that we fixed $\langle D_\delta \rangle_{\delta \in \mathrm{Lim}}$ a $\diamondsuit$-sequence at the start of this section.

Definition 7.46 (The $\checkmark$ relation)**.** *Let $\beta \in \gamma$, $k \leq l \in \omega$ and $\delta \in Lim$. Given $T \in Good_k(\beta)$ and $C \in FIN(\gamma)$, we say that (C, δ) accepts (T, β, k, l), and write it as $(C, \delta)\checkmark(T, \beta, k, l)$, if the following conditions hold:*

(1) $C \subseteq D_\delta$ *and* C *is captured at level* l,

(2) $T\bigstar(\beta,\, C(0),\, k,\, l-1)$,.

(3) $|(\beta)_{l-1} \cap \delta| = r_l$. *Equivalently, if* $F \in \mathcal{F}_l$ *is such that* $C \subseteq F$, *then* $(\beta)_{l-1} \cap \delta = R(F)$.

Whenever there is C for which $(C, \delta)\checkmark(T, \beta, k, l)$, such C has cardinality at least 1 and at most n_l due to the point (1). Thus, we can define $j(\delta, T, \beta, k, l)$ as the maximum of such cardinalities. That is:

$$j(\delta, T, \beta, k, l) = \max(\, j \leq n_l \,:\, \exists C \in [\gamma]^j \,(\,(C, \delta)\checkmark(T, \beta, k, l)\,)\,).$$

If there is no such C, then we define $j(\delta, T, \beta, k, l)$ as 0.

Definition 7.47 (The IH_2 property)**.** *Given $\delta \in Lim \cap \gamma$, we say that $IH_2(\delta, \mathcal{F})$ holds if one of the two following mutually excluding conditions occurs:*

(A) *There are infinitely many $l \in \omega$ for which there is $C \in FIN(D_\delta)$ which is fully captured at level l. In this case, we will say that $IH_2^A(\delta, \mathcal{F})$ holds.*

(B) *For all $\beta \in [\delta, \gamma)$, $2 \leq k \in \omega$ and $T \in Good_k(\beta)$ there are infinitely many $k \leq l \in \omega$ for which $j = j(\delta, T, \beta, k, l) < n_l$ and such that:*

  (B.1) $|(\beta)_{l-1} \cap \delta| = r_l$,

  (B.2) $\Xi_\beta(l) = j$,

  (B.3) *If $j > 0$ then there is $C \in [\delta]^j$ for which $(C, \delta)\checkmark(T, \beta, k, l)$ and $C \subseteq (\beta)_l$.*

  In this case we will say that $IH_2^B(\delta, \mathcal{F})$ holds.

Finally, we say that $\mathcal{F}$ satisfies IH_2 if $IH_2(\delta, \mathcal{F})$ holds for any $\delta \in Lim \cap \gamma$.

The two following results follow directly from the definition.

Proposition 7.48. *$\mathcal{F}(\omega)$ satisfies IH_2.*

Lemma 7.49. *Assume that γ is a limit of limit ordinals and for each limit $\gamma' < \gamma$, $\mathcal{F}|_{\gamma'}$ satisfies IH_2. Then $\mathcal{F}$ satisfies IH_2.*

More important, the property IH_2 in fact implies full capturing when $\gamma = \omega_1$.

Theorem 7.50. *Suppose that $\gamma = \omega_1$ and $\mathcal{F}$ satisfies IH_2. Then $\mathcal{F}$ is fully capturing.*

Proof. We will prove the Theorem by appealing to the equivalence of fully capturing stated in Lemma 5.14. Let $S \in [\omega_1]^{\omega_1}$. Since $\langle D_\delta \rangle_{\delta \in \mathrm{Lim}}$ is a $\diamondsuit$-sequence, there is $\delta \in \mathrm{Lim}$ so that:

(1) $S \cap \delta = D_\delta$.

(2) $(\delta, <, D_\delta, \mathcal{F}|_\delta)$ is an elementary submodel of $(\omega_1, <, S, \mathcal{F})$.

Suppose towards a contradiction that there is no $C \in \mathrm{FIN}(S)$ so that C is fully captured. By elementary, the same is true for D_δ. In other words, $IH_2^A(\delta, \mathcal{F})$ fails and consequently $IH_2^B(\delta, \mathcal{F})$ holds. Let us fix $\beta \in S \backslash \delta$ and $k = 2$. We now define $\mathbb{I} = \emptyset$, $z = \|\beta\|_2$ and $T = \{(\mathbb{I}, z)\} \in Good_2(\beta)$. According to the point (B) in Definition 7.47, there is $2 \leq l \in \omega$ for which $j = j(\delta, T, \beta, 2, l) < n_l$ and such that:

- $|(\beta)_{l-1} \cap \delta| = r_l$,
- $\Xi_\beta(l) = j$,
- If $j > 0$ then there is $C \in [\delta]^j$ for which $(C, \delta) \checkmark (T, \beta, 2, l)$ and $C \subseteq (\beta)_l$.

We will arrive to the desired contradiction by finding a finite set $C' \in \mathrm{FIN}(\omega_1)$ for which $(C', \delta) \checkmark (T, \beta, 2, l)$ and $|C'| = j + 1$. The proof of this is divided into two cases:

Case 1: If $j = 0$.

Proof of case. In this case, as $\Xi_\beta(l) = 0$, it is straightforward that $\{\beta\}$ is captured at level l. By elementarity, there is $\xi \in D_\delta$ so that $\|\xi\|_l = \|\beta\|_l$. In particular, $\{\xi\}$ is captured at level l. The proof of this case ends by proving the following claim.

Claim 1: $(\{\xi\}, \delta) \checkmark (T, \beta, k, l)$.

Proof of claim. It is enough to show that $T \bigstar (\beta, \xi, 2, l - 1)$. For this, note that the point (a) of Definition 7.43 is satisfied for the unique pair $(\mathbb{I}, z) \in T$ because $\|\beta\|_l = \|\xi\|_l$. In particular, $\Delta(\beta, \xi) > l$ so $\|\xi\|_2 = \|\beta\|_2 = z$. The point (b) of Definition 7.43 holds because $\mathbb{I} = \emptyset$. Therefore, $\pi_\xi^{2,l} = \emptyset \in Bl(0, m_l)$. Finally, the point (c) of Definition 7.43 is satisfied because $(\xi)_l(\|\beta\|_l) = (\xi)_l(\|\xi\|_l) = \xi$. □

□

Case 2: If $j > 0$.

Proof of case. Let $C \in [\delta]^j$ be such that $(C, \delta) \checkmark (T, \beta, 2, l)$ and $C \subseteq (\beta)_l$. By elementarity, there is $\xi \in D_\delta$ so that $\|\xi\|_l = \|\beta\|_l$ and $C \subseteq (\xi)_l$. Particularly, $\Xi_\xi(l) = j$ and $\rho(C(i), \xi) \leq l$ for each $i < j$. In fact, as $\Xi_{C(i)}(l) = i$ because C is captured at level l, we have that $\rho(C(i), \xi) = l$ due to Lemma 3.31. We will finish the proof of this case by showing the next claim.

Claim 2: $(C \cup \{\xi\}, \delta) \checkmark (T, \beta, k, l)$.

Proof of claim. It suffices to prove that $C \cup \{\xi\}$ is captured at level l. For this, we appeal to the Proposition 5.10. By the previous observations, we only need to prove that $\Delta(C(i), \xi) = \rho(C(i), \xi)$ for each $i < j$. First note that, according to the point (2) in Definition 7.46, $T \bigstar (\beta, C(0), 2, l - 1)$. As $(\emptyset, \|\beta\|_2)$ is the only element of T, then $\|\beta\|_{l-1} \leq \|C(0)\|_{l-1}$, $\|C(0)\|_2 = \|\beta\|_2$ and $\|\beta\|_{l-1} \in (\|C(0)\|_{l-1})_2$. In other words, $\|\beta\|_{l-1}$ is in the domain of $\phi_2^{\|C(0)\|_{l-1}}$. Moreover,

$$\begin{aligned} \phi_2^{\|C(0)\|_{l-1}}(\|C(0)\|_{l-1}) = \| \, \|C(0)\|_{l-1} \, \|_2 = \|C(0)\|_2 \\ = \|\beta\|_2 = \| \, \|\beta\|_{l-1} \, \|_2 = \phi_2^{\|C(0)\|_{l-1}}(\|\beta\|_{l-1}) \end{aligned}$$

due to the part (a) of Proposition 7.13. Since $\phi_2^{\|C(0)\|_l}$ is one to one, it follows that $\|C(0)\|_{l-1} = \|\beta\|_{l-1}$. Now, as $\|\beta\|_l = \|\xi\|_l$ and $\|C(0)\|_{l-1} = \|C(i)\|_{l-1}$, then $\|\xi\|_{l-1} = \|C(i)\|_{l-1}$. That is, $l \leq \Delta(C(i),\xi) \leq \rho(C(i),\xi) \leq l$. This finishes the proof. $\square$

$\square$

$\square$

We now present the key concept needed for proving Theorem 7.60.

Definition 7.51 (Transferring Good sets)**.** *Let $\beta \in \gamma$, $k \in \omega$ and $T \in Good_k(\beta)$. Given $k' > k$, we define $\mathcal{Q}_{k'}(\beta, T)$ as the set of all pairs $(F, \mathbb{I}) \in \mathcal{F}_k(m_{k'}) \times Bl_k(\beta)$ such that:*

- *There is $z \in m_k$ such that $(\mathbb{I}, z) \in T$,*
- *$\|\beta\|_{k'} \in F$,*
- *$F \circ \mathbb{I} \in Bl_{k'}(\beta)$. In other words, $F[\mathbb{I}(i)]$ is an interval in m'_k for every $i < \mathbb{I}$.*

Given $\alpha \in (\beta)^-_{k'}$ and $(F, \mathbb{I}) \in \mathcal{Q}_{k'}(\beta, T)$, let $\mathbb{I}_F = (\|\alpha\|_{k'}, \|\beta, \|_{k'}]^\frown(F \circ \mathbb{I})$[17]*. That is, $\mathbb{I}_F = \langle \mathbb{I}_F(i) \rangle_{i<\mathbb{I}+1}$ is given by:*

$$\mathbb{I}_F(i) = \begin{cases} (\|\alpha\|_{k'}, \|\beta\|_{k'}] & \text{if } i = 0 \\ F[\mathbb{I}(i-1)] & \text{if } i > 0 \end{cases}$$

We define the transferring of T relative to k', α and β as:

$$Transf_{k'}(T, \alpha, \beta) = \{\, (\mathbb{I}_F,\ F(z)) \,:\, (\mathbb{I}, z) \in T \text{ and } (F, \mathbb{I}) \in \mathcal{Q}_{k'}(\beta, T) \,\}$$

Remark 7.52. *Note that if $T \in Good_k(\beta)$, $\alpha \in (\beta)^-_k$ and $k' > k$ then $Transf_{k'}(T, \alpha, \beta) \in Good_{k'}(\alpha)$.*

Lemma 7.53. *Let $\beta \in \gamma$, $k \in \omega$, $T \in Good_k(\beta)$, $k' > k$ and $\alpha \in (\beta)^-_{k'}$. Suppose that $l > k'$ and $\xi \in \gamma$ are such that:*

1. *$\|\beta\|_l \leq \|\xi\|_l$,*
2. *$(\xi)_l(\|\beta\|_l) \in (\xi)_{k'}$. In other words, $\|\beta\|_l \in (\|\xi\|_l)_{k'}$*
3. *$(\beta)_{k'}[\,(\|\alpha\|_{k'}, \|\beta\|_{k'}]\,] = (\beta)_{k'} \backslash (\alpha+1)$ is an interval in $(\beta)_l$.*

Then the following statements are equivalent:

- (a) *$T \bigstar (\beta, \xi, k, l)$.*
- (b) *$Transf_{k'}(T, \alpha, \beta) \bigstar (\alpha, \xi, k', l)$.*

Proof. Let $\mathcal{Q} = \mathcal{Q}_{k'}(\beta, T)$ and $T' = \text{Transf}_{k'}(T, \alpha, \beta)$.

$(a) \Rightarrow (b)$. Let $(\mathbb{I}, z) \in T$ so that $\|\beta\|_k \leq \|\xi\|_k = z$, $\pi^{k,l}_\xi \circ \mathbb{I} \in Bl(0, m_l)$ and $\|\beta\|_l \in (\|\xi\|_l)_k$. Now, let us take $F \in \mathcal{F}_k(m_{k'})$ having $\|\xi\|_{k'}$ as an element. Before we continue, we will highlight some useful facts:

- $\|\xi\|_{k'} \in F$ so $F \cap (\xi+1) = (\,\|\xi\|_{k'})_k$. Consequently, $F(a) = (\phi_k^{\|\xi\|_{k'}})^{-1}(a)$ for each $a \leq \|\xi\|_k$.
- $\|\xi\|_k = \phi_k^{(\|\xi\|_{k'})} \circ \phi^\xi_k(\xi)$ according to Lemma 7.14. Therefore, $F(\|\xi\|_k) = \|\xi\|_{k'}$ by virtue of the previous point.
- Again, by Lemma 7.14:

$$\phi_{k'}^{\|\xi\|_l} \circ \pi^{k,l}_\xi = \phi_{k'}^{\|\xi\|_l} \circ \phi^\xi_l \circ (\phi^\xi_k)^{-1} = \phi^\xi_{k'} \circ (\phi^\xi_k)^{-1} = (\phi_k^{\|\xi\|_{k'}})^{-1}.$$

[17]In principle, $\mathbb{I}_F$ also depends on α. However, we will never use this concept with two different α's at the same time. For that reason, we omit any mention of that ordinal in the notation.

Claim 1: $(F, \mathbb{I}) \in \mathcal{Q}$.

Proof of claim. First we will show that $\|\beta\|_{k'} \in F$. Indeed, since $k \leq k' < l$ and $\|\beta\|_l \in (\|\xi\|_l)_k$ then, by Proposition 7.13 and Lemma 7.14, we can conclude that

$$\begin{aligned}\|\beta\|_{k'} = \| \, \|\beta\|_l \, \|_{k'} = \phi_{k'}^{\|\xi\|_l}(\|\beta\|_l) \in \phi_{k'}^{\|\xi\|_l}[\, (\, \|\xi\|_l \,)_k \,] \\ = (\, \phi_{k'}^{\|\xi\|_l}(\|\xi\|_l) \,)_k \\ = (\, \| \, \|\xi\|_l \, \|_{k'} \,)_k = (\|\xi\|_{k'})_k \subseteq F.\end{aligned}$$

Now, let $i < |\mathbb{I}|$. The claim will be over once we prove that $F[\mathbb{I}(i)]$ is an interval in $m_{k'}$. As $\mathbb{I}(i) \leq z = \|\xi\|_k$ then $F[\mathbb{I}(i)] = (\phi_k^{\|\xi\|_{k'}})^{-1}[\mathbb{I}(i)] = \phi_{k'}^{\|\xi\|_l} \circ \pi_\xi^{k,l}[\mathbb{I}(i)]$. By virtue of the hypotheses, $\pi_\xi^{k,l}[\mathbb{I}]$ is an interval in m_l which is contained in $(\, \|\xi\|_l \,)_k$. Hence, such interval is also included in $(\, \|\xi\|_l \,)_{k'}$. Since, $\phi_{k'}^{\|\xi\|_l}$ is the increasing bijection from $(\, \|\xi\|_l \,)_{k'}$ to $\|\xi\|_{k'}$, it follows that $\phi_{k'}^{\|\xi\|_l}[\pi_\xi^{k,l}[\mathbb{I}]]$ is an interval in $\|\xi\|_{k'} + 1$. From this it is straightforward that $F[\mathbb{I}(i)]$ is an interval in $m_{k'}$. This finishes the claim. □

Since $(\mathbb{I}, z) \in T$ and $(F, \mathbb{I}) \in \mathcal{Q}$ then $(\mathbb{I}_F, F(z)) \in T'$. We will show that such pair testifies that $T' \bigstar (\alpha, \xi, k', l)$. This will be achieved by proving the three following claims.

Claim 2: $\|\alpha\|_l \leq \|\xi\|_l$ and $\|\xi\|_{k'} = F(z)$.

Proof of claim. We know that $z = \|\xi\|_k$, and it has already been proved that $F(\|\xi\|_k) = \|\xi\|_{k'}$. As $\alpha \in (\beta)_{k'}$ then $\|\alpha\|_l \leq \|\beta\|_l$. But we know by the hypotheses that $\|\beta\|_l \leq \|\xi\|_l$. So we are done

□

Claim 3: $\pi_\xi^{k',l} \circ \mathbb{I}_F \in Bl(0, m_l)$.

Proof of claim. Let $i < |\mathbb{I}_F|$. We want to show that $\pi_\xi^{k',l}[\mathbb{I}_F(i)]$ is an interval in m_l. First we deal with the case where $i = 0$. Here, $\mathbb{I}_F(i) = (\|\alpha\|_{k'}, \|\beta\|_{k'}]$. We will first argue that, in this case, $\pi_\xi^{k',l}[\mathbb{I}_F(i)] = \pi_\beta^{k',l}[\mathbb{I}_F(i)]$. Indeed, since $\|\beta\|_l \in (\|\xi\|_l)_{k'}$ then $\phi_{k'}^{\|\xi\|_l}|_{\|\beta\|_l+1} = \phi_{k'}^{\|\beta\|_l}$. In particular, this means that $(\phi_{k'}^{\|\xi\|_l})^{-1}[a] = (\phi_{k'}^{\|\beta\|_l})^{-1}[a]$ for each $a \leq \|\beta\|_{k'}$. In this way, by using Lemma 7.14, we conclude that

$$\begin{aligned}\pi_\xi^{k',l}[\mathbb{I}_F(i)] &= \phi_l^\xi \circ (\phi_{k'}^\xi)^{-1}[\mathbb{I}_F(i)] \\ &= \phi_l^\xi \circ ((\phi_l^\xi)^{-1} \circ (\phi_{k'}^{\|\xi\|_l})^{-1})[\mathbb{I}_F(i)] \\ &= (\phi_{k'}^{\|\beta\|_l})^{-1}[\mathbb{I}_F(i)] = (\phi_{k'}^{\|\beta\|_l})^{-1}[\mathbb{I}_F(i)] \\ &= (\phi_l^\beta \circ ((\phi_l^\beta)^{-1}) \circ (\phi_{k'}^{\|\beta\|_l})^{-1}[\mathbb{I}_F(i)] \\ &= \phi_l^\beta \circ (\phi_{k'}^\beta)^{-1}[\mathbb{I}_F(i)] = \pi_\beta^{k',l}[\mathbb{I}_F(i)]\end{aligned}$$

Observe that $(\phi_{k'}^\beta)^{-1}[\mathbb{I}(i)] = (\beta)_{k'}[\, (\|\alpha\|_{k'}, \|\beta\|_{k'}] \,]$ is an interval in $(\beta)_l$, according to the point (3) in the hypotheses of this lemma. Since ϕ_l^β is the increasing bijection from $(\beta)_l$ to $\|\beta\|_l + 1 \sqsubseteq m_l$, then $\pi_\beta^{k',l}[\mathbb{I}(i)]$ is an interval in m_l. This finishes the first case. Now we deal with the case where $i > 0$. Here, $\mathbb{I}_F(i) = F[\mathbb{I}(i-1)]$. Thus,

$$\begin{aligned}
\pi_\xi^{k',l}[\mathbb{I}_F(i)] &= \phi_l^\xi \circ (\phi_{k'}^\xi)^{-1}[F(i)] \\
&= \phi_l^\xi \circ (\phi_{k'}^\xi)^{-1}[(\phi_k^{\|\xi\|_{k'}})^{-1}[\mathbb{I}(i-1)]] \\
&= \phi_l^\xi \circ ((\phi_{k'}^\xi)^{-1} \circ (\phi_k^{\|\xi\|_{k'}})^{-1})[\mathbb{I}(i-1)] \\
&= \phi_l^\xi \circ (\phi_k^\xi)^{-1}[\mathbb{I}(i-1)] = \pi_\xi^{k,l}[\mathbb{I}(i-1)]
\end{aligned}$$

Due to the hypotheses, $\pi_\xi^{k,l}[\mathbb{I}(i-1)]$ is already an interval in m_l. Consequently, this case is over. □

Claim 4: $\|\alpha\|_l \in (\|\xi\|_l)_{k'}$.

Proof of claim. Recall that $\alpha \in (\beta)_k$. Therefore, $\|\alpha\|_l \in (\|\beta\|_l)_k$ due to the point (b) of Proposition 7.13. As $\|\beta\|_l \in (\|\xi\|_l)_{k'}$, we are done. □

□

(b) $\Rightarrow$ *(a)*. Let $(\mathbb{I}, z) \in T$ and $F \in \mathcal{F}_k(m_{k'})$ with $(F, \mathbb{I}) \in \mathcal{Q}$ be such that $\|\alpha\|_{k'} \leq \|\xi\|_{k'} = F(z)$, $\pi_\xi^{k',l} \circ \mathbb{I}_F \in Bl(0, m_l)$ and $\|\alpha\|_l \in (\|\xi\|_l)_{k'}$. We will show in the next three claims that $(\mathbb{I}, z)$ testifies that $T\bigstar(\beta, \xi, k', l)$.

Claim 5: $\|\beta\|_l \leq \|\xi\|_l$ and $\|\xi\|_k = z$.

Proof of claim. Due to the hypotheses of the lemma, we already know that $\|\beta\|_l \leq \|\xi\|_l$. Therefore it suffices to prove that $\|\xi\|_k = z$. Indeed, as $F \in \mathcal{F}_k(m_{k'})$ then $\|F(z)\|_k = z$. Moreover, we know that $F(z) = \|\xi\|_{k'}$. Thus, $z = \|F(z)\|_k = \| \, \|\xi\|_{k'} \, \|_k = \|\xi\|_k$ by virtue of the point (a) in Proposition 7.13. □

Claim 6: $\pi_\xi^{k,l} \circ \mathbb{I} \in Bl(0, m_l)$.

Proof of claim. Let $i < |\mathbb{I}|$. We need to prove that $\pi_\xi^{k,l}[\mathbb{I}(i)]$ is an interval in m_l. First note that $\|\xi\|_{k'} \in F$, so as in the proof of the other direction of the lemma, $F \cap (\xi+1) = (\|\xi\|_{k'})_k$. In this way, $F(a) = (\phi_k^{\|\xi\|_{k'}})^{-1}(a)$ for each $a \leq \|\xi\|_k$. From this fact, we get the following chain of equalities:

$$\begin{aligned}
\pi_\xi^{k,l}[\mathbb{I}(i)] = \phi_l^\xi \circ (\phi_k^\xi)^{-1}[\mathbb{I}(i)] &= \phi_l^\xi \circ ((\phi_{k'}^\xi)^{-1} \circ (\phi_k^{\|\xi\|_{k'}})^{-1})[\mathbb{I}(i)] \\
&= \phi_l^\xi \circ \phi_{k'}^\xi[F[\mathbb{I}(i)]] \\
&= \pi_\xi^{k',l}[\mathbb{I}_F(i+1)]
\end{aligned}$$

As $\pi_\xi^{k',l} \circ \mathbb{I} \in Bl(0, m_l)$, we are done. □

Claim 7: $\|\beta\|_l \in (\|\xi\|_l)_k$.

Proof of claim. On one hand, $\|\beta\|_{k'} \in F$ since $(F, \mathbb{I}) \in \mathcal{Q}$. On the other hand, we already know that $\|\xi\|_k \in F$. Therefore, $\rho(\|\beta\|_{k'}, \|\xi\|_{k'}) \leq k$. Now, according to the hypotheses $\rho(\|\xi\|_l, \|\beta\|_l) \leq k'$. Thus,

$$\rho(\|\beta\|_l, \|\xi\|_l) = \rho(\| \, \|\beta\|_l \, \|_{k'}, \| \, \|\xi\|_l \, \|_{k'}) = \rho(\|\beta\|_{k'}, |\xi\|_{k'})$$

by virtue of the points (a) and (b) of Proposition 7.13. In this way, $\rho(\|\beta\|_l, \|\xi\|_l) \leq k$.

□

□

□

Corollary 7.54. *Let $\mathcal{F}$ be a construction scheme over an ordinal γ. Also, let $\beta \in \gamma$, $T \in Good_k(\beta)$, $k' > k$ and $\alpha \in (\beta)^-_{k'}$. Suppose that $(\beta)_{k'} \backslash (\alpha+1)$ is an interval in $(\beta)_{l-1}$. Furthermore, assume that $l > k'$ and $\delta \in Lim \cap \alpha$ are such that $|(\alpha)_{l-1} \cap \delta| = r_l$. If $C \in FIN(\gamma)$ is such that $(C,\delta)\checkmark(k,l,\beta,\delta,T)$ then $(C,\delta)\checkmark(k',l,\beta,\delta,T')$ where $T' = Transf_{k'}(\alpha,\beta,T)$. In particular, $j(\delta,T,\beta,k,l) \leq j(\delta,T',\alpha,k',l)$.*

Proof. Let $C \in \mathrm{FIN}(\gamma)$ be such that $(C,\delta)\checkmark(k,l,\beta,\delta,T')$. Since $\alpha \in (\beta)_{k'}$ and $l > k'$ then $(\beta)_l \cap (\alpha+1) = (\alpha)_l$. By the hypotheses, $\alpha \geq \delta$. This means that $(C(0))_l[r_l] = (\beta)_l \cap \delta = (\alpha)_l \cap \delta$. The only thing left to prove is that $T'\bigstar(\alpha, C(0), k', l-1)$. For this, it suffices to show that the hypotheses of Lemma 7.53 hold for $\xi = C(0)$. Indeed, since $T\bigstar(\beta,\xi,k,l-1)$ then $\|\beta\|_{l-1} \in (\|\xi\|_{l-1})_k$. This implies both that $\|\beta\|_{l-1} \leq \|\xi\|_{l-1}$ and $\|\beta\|_{l-1} \in (\|\xi\|_{l-1})_{k'}$. According to the assumptions $(\beta)_{k'}[\,(\|\alpha\|_{k'},\, \|\beta\|_{k'}]\,]$ is an interval in $(\beta)_{l-1}$. Thus, we are done.

□

We now aim to prove that if γ is a countable ordinal and $\mathcal{F}$ satisfies IH_1 and IH_2, then we can extend $\mathcal{F}$ to a construction scheme over $\gamma+\omega$ which also satisfies IH_1 and IH_2. In subsection 7.1 we already proved that there is a countable family of dense sets over $\mathbb{P}(\mathcal{F})$ so that if G is a filter intersecting each of those sets, then $\mathcal{F}^{\mathcal{G}}$ is construction scheme over $\gamma+\omega$ which also satisfies IH_1. Here we want to show the same result but for the Property IH_2. However, due to the large amount of variables appearing in the Definition 7.47, the explicit formula for each of those dense sets is quite messy. For that reason, we will instead show that $\mathbb{P}(\mathcal{F})$ forces $\mathcal{F}^{Gen}$ to satisfy IH_2. Before doing this, it is worth pointing out some last remarks regarding the forcing $\mathbb{P}(\mathcal{F})$.

Remark 7.55. *By virtue of the the Lemmas 7.31,and 7.33, we have that*

$$\mathbb{P}(\mathcal{F}) \Vdash \text{“}\mathcal{F}^{Gen} \text{ is construction scheme over } \gamma+\omega \text{ which satisfies } IH_1\text{”}.$$

Remark 7.56. *$\mathbb{P}(\mathcal{F}) \Vdash$ “$\mathcal{F} \subseteq \mathcal{F}^{Gen}$” due to the Lemma 7.32. This means that $\rho_{F^{Gen}}|_{\gamma^2}$ is forced to be equal to $\rho_{\mathcal{F}}$. Therefore, there is no risk of confusion by referring to both of these two ordinal metrics simply as ρ.*

Remark 7.57. *If $p \in \mathbb{P}_k(\mathcal{F})$ for some $k \in \omega$, then $p \Vdash$ “$p \in \mathcal{F}^{Gen}_k$ ”. In particular, $\mathcal{F}(p)$ is forced to be equal to $\mathcal{F}^{Gen}|_p$. In other words, $\rho|_{p^2} = \rho_{\mathcal{F}(p)}$. So, again, there is no risk of confusion on referring to $\rho_{\mathcal{F}(p)}$, simply as ρ.*

Remark 7.58. *Suppose that $p \in \mathbb{P}_l(\mathcal{F})$, $\beta \in p$ and $k \leq l$. Then for each $H \in \mathcal{F}_j(p)$ it happens that $p \Vdash$ “$H \cap (\beta+1) = (\beta)_k$ ”.[18] That is, p decides the value of $(\beta)_k$. Furthermore, this value can be calculated using $\mathcal{F}(p)$. From this, it follows that p also knows who is $\|\beta\|_k$, $\Xi_\beta(k)$, $Bl_k(\beta)$ $(= Bl(\|\beta\|_k, m_k))$ and $Good_k(\beta)$. Therefore, there should not be any confusion while working explicitly with this sets instead of with their names, of course, as long as the conditions at the beginning of this remark hold.*

Lemma 7.59. *Let $q \in \mathbb{P}(\mathcal{F})$. Then there is $p \leq q$ so that $|p \cap \gamma| = r_{k_p+1}$.*

[18]Here, by $(\beta)_k$ we mean k-closure of β calculated inside $\gamma+\omega$ with respect to $\rho_{\mathcal{F}^{Gen}}$.

Proof. From Lemma 7.31, it is easy to get $q' \leq q$ so that $k_{q'} > k_q$. Now, according to Lemma 7.33, there is $p' \in \mathbb{P}(\mathcal{F})$ so that $p' \leq q'$ and $p' \cap \gamma = R(p')$. Put $p = p'_0$. Then $p \in \mathbb{P}(\mathcal{F})$ and $p \leq q$. Furthermore, $|p \cap \gamma| = |R(p')| = r_{k_{p'}} = r_{k_p+1}$. $\square$

Theorem 7.60. *Suppose that γ is countable and $\mathcal{F}$ satisfies IH_1 and IH_2. Then*

$$\mathbb{P}(\mathcal{F}) \Vdash \text{“ } \mathcal{F}^{Gen} \textit{ satisfies } IH_2 \text{ ”}.$$

Proof. According to the Definition 7.47, our goal is to prove that

$$\mathbb{P}(\mathcal{F}) \Vdash \text{“ } IH_2(\delta, \mathcal{F}^{Gen}) \text{ holds ”}$$

for any $\delta \in \mathrm{Lim} \cap (\gamma + \omega)$. If $\mathcal{F}$ satisfies that there are infinitely many $l \in \omega$ for which there is $C \in \mathrm{FIN}(D_\delta)$ which is fully captured at level l, it is easy to see that $\mathbb{P}(\mathcal{F}) \vDash$ “ $IH_2^A(\gamma, \mathcal{F}^{Gen})$ holds ”. This is because $\mathbb{P}(\mathcal{F}) \Vdash$ “ $\mathcal{F} \subseteq \mathcal{F}^{Gen}$ ”. Therefore, the only interesting case happens when:

With respect to $\mathcal{F}$, there is $k'' \in \omega$ such that for each $l \geq k''$ there is no $C \in \mathrm{FIN}(D_\gamma)$ which is fully captured at level l.

From now on, let us assume that this case holds.

Claim: $\mathbb{P}(\mathcal{F}) \Vdash$ “ $IH_2^B(\delta, \mathcal{F}^{Gen})$ holds ”.

Proof of claim. Let $q \in \mathbb{P}(\mathcal{F})$, $\beta \in [\delta, \gamma + \omega)$, $2 \leq k \in \omega$ and a name $\dot{T}$ for an element of $Good_k(\beta)$. Because of Lemma 7.31 , we may assume without loss of generality that $\beta \in q$ and $q \in \mathbb{P}_{k'}(\mathcal{F})$ for some $k' > k, k''$. According to the Remark 7.58, $Good_k(\beta)$ is fully determined by q. In particular, this means that there is T such that $q \Vdash$ “ $T = \dot{T}$ ”.

We need to show that there is $l > k'$ and $p \leq q$ with the following properties:

- $p \Vdash$ “ $|(\beta)_{l-1} \cap \delta| = r_l$ ”.
- There is $j \in \omega$ so that $p \Vdash$ “ $j = j(\delta, T, \beta, k, l)$ ”.
- $p \Vdash$ “ $\Xi_\beta(l) = j$ ”.
- If $j > 0$ then there is $C \in [\gamma + \omega]^j$ so that $p \Vdash$ “ $(C, \delta)\checkmark(T, \beta, k, l)$ and $C \subseteq (\beta)_l$”.

We will divide the rest of the proof into two cases.

Case 1: $\delta = \gamma$.

Proof of case. By virtue of the Lemma 7.59, there is $q' \in \mathbb{P}(\mathcal{F})$ so that $q' \leq q$ and $|q' \cap \gamma| = r_{k_q+1}$. Let $l = k_q + 1$. Note that the objects needed to calculate $j(\gamma, T, \beta, k, l)$ are $\mathcal{F}^{Gen}|_\gamma$, T, $(\beta)_{l-1}$, k an l. All of this objects are already decided by q'. Therefore, there is $j \in \omega$ so that $q' \Vdash$ “ $j = j(\gamma, T, \beta, k, l)$”.

Subcase 1.1: If $j = 0$.

Proof of subcase. Consider $F \in \mathcal{F}_l$ so that $q' \cap \gamma \sqsubseteq F$. Then $R(F) = q' \cap \gamma$. Let $\alpha = \min(F_0 \backslash R(F))$ and $p = Cut_\alpha(F)$. It is straightforward that $R(p) = q' \cap \gamma$ and $q' = p_0$. This means that $p \leq q$. Moreover, as $\beta \in q \backslash \gamma$ then $p \Vdash$ “$|(\beta)_{l-1} \cap \delta| = r_l$ ” and $p \Vdash$ “ $\Xi_\beta(l) = 0 = j$ ”. In this case, there is nothing more to prove. $\square$

Subcase 1.2: If $j > 0$.

Proof of subcase. By definition of j, there is $C \in [\gamma+\omega]^j$ so that $(C,\gamma)\checkmark(T,\beta,k,l)$. According the point (1) of Definition 7.46, $C \subseteq D_\gamma$ and C is captured at level l. Let $F \in \mathcal{F}_l$ be so that $C \subseteq F$. Then, by the point (3) of Definition 7.46, it follows that $R(F) = (\beta)_{l-1} \cap \gamma = q' \cap \gamma$. In this case, let $\alpha = \min(F_j \backslash R(F))$ and $p = Cut_\alpha(F)$. As in the previous case, we have $p_j = q'$ and $R(p) = q' \cap \gamma$. Thus, $q' \leq p_j$. Again, as $\beta \in q\backslash\gamma$ then $p \Vdash$ " $|(\beta)_{l-1} \cap \delta| = r_l$ " and $p \Vdash$ " $\Xi_\beta(l) = j$ ". In order to finish, note that $p \cap \gamma = \bigcup\limits_{i<j} F_i$. In particular, $C \subseteq p \cap \gamma \subseteq (\beta+1) \cap p = (\beta)_l$. That is,

$$p \Vdash \text{“ } (C,\gamma)\checkmark(T,\beta,k,l) \text{ and } C \subseteq (\beta)_l \text{ ”}.$$

□

□

Case 2: If $\delta < \gamma$.

Proof of case. If $\beta < \gamma$ there is nothing to do. This is because $\mathcal{F}$ is forced to be contained in $\mathcal{F}^{Gen}$ and $\mathcal{F}$ already satisfies IH_2. Therefore, we may assume that $\beta \geq \gamma$. Appealing to Lemma 7.31, we may assume without loss of generality that $q \cap [\delta,\gamma) \neq \emptyset$. Let $\alpha = \alpha_q = \max(q \cap \gamma) \geq \delta$. The plan is apply the property IH_2 to α (inside $\mathcal{F}$). For this purpose, let $\mathcal{Q} = \mathcal{Q}_{k'}(\beta, T)$ and $T' = \text{Transf}_{k'}(T,\alpha,\beta)$.

By our assumptions $IH_2^A(\delta,\mathcal{F})$ does not hold. Therefore, $IH_2^B(\delta,\mathcal{F})$ holds. This means that there is $l \geq k'$ for which $j' = j(\delta,T',\alpha,k',l) < n_l$ and such that:

(I) $|(\alpha)_{l-1} \cap \delta| = r_l$,
(II) $\Xi_\alpha(l) = j$,
(III) If $j > 0$ then there is $C \in [\gamma]^j$ for which $(C,\delta)\checkmark(T',\alpha,k',l)$ and $C \subseteq (\alpha)_l$.

Before we continue, we remark that:

- $q \Vdash$ " $\alpha \in \beta \cap q = (\beta)^-_{k'}$ ".
- $q \Vdash$ " $(\beta)_{k'}\backslash(\alpha+1) = [\gamma,\beta+1)$ ". Since $[\gamma,\beta+1)$ is already an interval in $\gamma+\omega$, it follows that $q \Vdash$ " $(\beta)_{k'}$ is an interval in $(\beta)_{l-1}$".
- $q \Vdash$ " $(\beta)^-_{l-1} \cap \alpha = (\alpha)_{l-1}$ " because $l > k'$. In particular, this means that $q \Vdash$ " $|(\beta)_{l-1} \cap \delta| = |(\alpha)_{l-1} \cap \delta| = r_l$ ".
- Since $j' \geq 0$ and $\Xi_\alpha(l) = j'$ then $q \Vdash$ " $\Xi_\beta(l) = j'$ " by means of the Lemma 3.31.

Due to the first two points, the hypotheses of Corollary 7.54 are fulfilled. In particular, this means that

$$q \Vdash \text{“}j(\delta,T,\beta,k,l) \leq j' \text{ ”}.$$

We will divide the rest of the proof into two subcases.

Subcase 2.1: If $j' = 0$.

Proof of subclaim. As $j(\delta,T,\beta,k,l)$ is always non-negative then

$$q \Vdash \text{“}j(\delta,T,\beta,k,l) = 0 \text{ ”}.$$

Because of this and by the last two points of the previous remark, $p = q$ forces everything that we want. □

Subcase 2.2: If $j' > 0$.

Proof of subcase. Let $F \in \mathcal{F}_l$ be such that $\alpha \in F$. Since $j > 0$, there is $C \in [\gamma]^j$ for which $(C,\delta)\checkmark(T',\alpha,k',l)$ and $C \subseteq (\alpha)_l$. Note that $q \vDash$ " $(\alpha)_l \subseteq (\beta)_l$". In this way, $q \Vdash$ " $C \subseteq (\beta)_l$ ". Our next task will be to extend q to a condition forcing that $T\bigstar(\beta, C(0), k, l-1)$. By appealing to Lemma 7.53, it suffices to find $p \leq q$ which forces that $\|\beta\|_{l-1} \leq \|C(0)\|_{l-1}$ and $\|\beta\|_{l-1} \in (\|\xi\|_{l-1})_{k'}$.

As $(C,\delta)\checkmark(T',\alpha,k',l)$ then $T'\bigstar(\alpha, C(0), k', l-1)$. Let $\xi \in F_j$ be such that $\|\xi\|_{l-1} = \|C(0)\|_{l-1}$. By Remark 7.44, $T'\bigstar(\alpha,\xi,k',l-1)$. Thus, there are $(\mathbb{I}, z) \in T$ and $(G, \mathcal{I}) \in \mathcal{Q}$ so that $(\mathbb{I}_G, G(z))$ testifies the previous guessing relation. That is:

(*a*) $\|\alpha\|_{l-1} \leq \|\xi\|_{l-1}$ and $\|\xi\|_{k'} = G(z)$. Since both ξ and α are in F_j then $\rho(\alpha,\xi) \leq l-1$. Thus, this point implies that $\alpha \leq \xi$.

(*b*) $(\xi)_{k'} \circ \mathbb{I}_G \in Bl(0, (\xi)_{l-1})$.

(*c*) $(\xi)_{l-1}(\|\alpha\|_{l-1}) \in (\xi)_{k'}$. As $\alpha, \xi \in F_j$ then $(\xi)_{l-1}(\|\alpha\|_{l-1}) = F_j(\|\alpha\|_{l-1}) = \alpha$. Therefore, this point implies that $\alpha \in (\xi)_{k'}$.

According to the point (b), $(\xi)_{k'}[\mathbb{I}_G(0)] = (\xi)_{k'}[\,(\|\alpha\|_{k'}, \|\beta\|_{k'}]]$ is an interval in $(\xi)_{l-1}$. Moreover, by the point (a), this interval is contained in $F_j \backslash R(F)$. Therefore, it is also an interval in F. Let $\alpha' = (\xi)_{k'}[\|\alpha\|_{k'} + 1]$. In other words, α' is the successor of α inside $(\xi)_{k'}$. Because of this, it follows that $(\alpha')^-_{k'} = (\alpha)_{k'} = q \cap \gamma$. In this way, $p = Cut_{\alpha'}(F) \leq q$ due to the Lemma 7.24.

<u>Subclaim 1</u>: $p \Vdash$ " $\|\beta\|_{l-1} \leq \|C(0)\|_{l-1}$ and $\|\beta\|_{l-1} \in (\|C(0)\|_{l-1})_{k'}$ ".

Proof of subclaim. Let $\phi : F \longrightarrow p$ be the increasing bijection. By the definition of the Cut function, it follows that $\phi(\alpha') = \gamma$. Now, since $(\xi)_{k'}[\mathbb{I}_F(0)]$ is an interval in F, then $\phi[\,(\xi)_{k'}[\mathbb{I}_G(0)]\,] = (\phi(\xi))_{k'}[\mathbb{I}_G(0)]$ is an interval in p. Furthermore, the first point of this interval is γ. In this way, as $p\backslash\gamma$ is an initial segment of $[\gamma, \gamma+\omega)$, we conclude that $\phi((\xi)'_k(\|\beta\|_{k'}))$ (the last point of the interval $\phi[(\xi)_{k'}[\,\mathbb{I}_G(0)]\,]$) is equal to

$$\gamma + (\|\beta\|_{k'} - \|\alpha\|_{k'}) = \beta.$$ [19]

As direct consequence of this equality, p forces that $\beta \leq \phi(\xi)$ and $\rho(\beta, \phi(\xi)) \leq k'$. It then follows that $\|\beta\|_{l-1} \leq \|\phi(\xi)\|_{l-1}$ and $\|\beta\|_{l-1} \in (\|\phi(\xi)\|_{l-1})_{k'}$. Since $\|\phi(\xi)\|_{l-1} = \|\xi\|_{l-1} = \|C(0)\|_{l-1}$, we are done. $\square$

By virtue of the previous subclaim, $p \Vdash$ " $T\bigstar(\beta, C(0), k, l-1)$ ". It then follows that $p \Vdash$ " $(C,\delta)\checkmark(\delta, T, \beta, k, l)$ ". As $|C| = j'$, we get that

$$p \Vdash \text{“ } j' \leq j(\delta, T, \beta, k, l) \text{ ”}.$$

Hence, these two numbers are equal. In conclusion, we have shown that there are $p \leq q$ and $j \in \omega$ (namely, j'), so that:

- $p \Vdash$ " $|(\beta)_{l-1} \cap \delta| = r_l$ " and $p \Vdash$ " $\Xi_\beta(l) = j$ ".[20]
- There is $C \in [\gamma+\omega]^j$ so that $p \Vdash$ " $(C,\delta)\checkmark(T,\beta,k,l)$ and $C \subseteq (\beta)_l$".

Thus, the proof of this subcase is over. $\square$

$\square$

$\square$

$\square$

[19] All the definitions related to IH_2 were originally formulated to achieve this single step of the proof.

[20] This was already forced by q.

It is not hard to see that the property forced in the previous theorem can be coded by countable many dense subsets of $\mathbb{P}(\mathcal{F})$. In this way, we have the following corollary.

Corollary 7.61. *Suppose that γ is a limit ordinal and $\mathcal{F}$ satisfies IH_1 and IH_2. There is a countable family of dense sets in $\mathbb{P}(\mathcal{F})$ so that if $\mathcal{G}$ is a filter intersecting all of them, then $\mathcal{F}^{\mathcal{G}}$ is a construction scheme over $\gamma+\omega$ which contain $\mathcal{F}$ and satisfies both IH_1 and IH_2.*

Now we have proved all the necessary results to show that there is a fully capturing construction scheme over ω_1 of type τ. Namely, Propositions 7.26 and 7.48 , Lemmas 7.27 and 7.49, and Corollary 7.61. From them we conclude that there is a sequence $\langle \mathcal{F}^\delta \rangle_{\delta\in \mathrm{Lim}}$ so that for all $\delta<\gamma\in \mathrm{Lim}$, the following properties hold:

- $\mathcal{F}_\delta$ is a construction scheme over δ satisfying IH_1 and IH_2,
- $\mathcal{F}_\delta \subseteq \mathcal{F}_\gamma$.

It follows that $\mathcal{F} = \bigcup\limits_{\delta\in\mathrm{Lim}} \mathcal{F}^\delta$ is a fully capturing construction scheme. This is due to the Theorem 7.50.

Theorem 7.62. *The $\Diamond$-principle implies FCA.*

Now fix $\mathcal{P}$ an aribtrary partition of ω compatible with τ. Consider the following variations of the the properties IH_1 and IH_2.

Definition 7.63 (The $IH_1(\mathcal{P})$ property)**.** *We say that $\mathcal{F}$ satisfies $IH_1(\mathcal{P})$ if for all $A \in$ FIN(γ), $P\in\mathcal{P}$ and $\alpha\in\gamma$, there is $F\in\mathcal{F}$ with $\rho^F\in P$ such that $IH_1(\alpha,A,F)$ holds.*

Definition 7.64 (The $IH_2(\mathcal{P})$ property)**.** *Given $\delta\in$ Lim $\cap\,\gamma$ and $P\in\mathcal{P}$, we say that $IH_2(\delta,\mathcal{F})$ holds if one of the two following mutually excluding conditions occurs:*

- (A) *There are infinitely many $l\in P$ for which there is $C\in$ FIN(D_δ) which is fully captured at level l. In this case, we will say that $IH_2^A(P,\delta,\mathcal{F})$ holds.*
- (B) *For all $\beta\in[\delta,\gamma)$, $2\le k\in\omega$ and $T\in$ Good$_k(\beta)$ there are infinitely many $k\le l\in P$ for which $j=j(\delta,T,\beta,k,l)<n_l$ and such that:*
  - $(B.1)$ $|(\beta)_{l-1}\cap\delta| = r_l$,
  - $(B.2)$ $\Xi_\beta(l)=j$,
  - $(B.3)$ *If $j>0$ then there is $C\in[\delta]^j$ for which $(C,\delta)\checkmark(T,\beta,k,l)$ and $C\subseteq(\beta)_l$.*

  In this case we will say that $IH_2^B(P,\delta,\mathcal{F})$ holds.

Finally, we say that $\mathcal{F}$ satisfies $IH_2(\mathcal{P})$ if $IH_2(\delta,\mathcal{F})$ holds for any $\delta\in$ Lim $\cap\,\gamma$ and each $P\in\mathcal{P}$.

All the results stated in this section are true when IH_1 is changed by $IH_1(\mathcal{P})$ and IH_2 is changed by $IH_2(\mathcal{P})$. Moreover, the proofs are completely similar. In this way, we deduce:

Theorem 7.65. *The $\Diamond$-principle implies FCA(part).*

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